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## Asset Return Predictability in a Heterogeneous Agent Equilibrium Model

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We use a general equilibrium model as a laboratory for generating predictable excess returns and for assessing the properties of the estimated consumption/portfolio rules, under both the empirical and the true dynamics of excess returns. The advantage of this approach, relative to the existing literature, is that the equilibrium model delineates the precise nature of the risk/return trade-off within an optimizing setting

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that endogenizes return predictability. In the experiments that we consider, the estimation issues are so severe that simple unconditional consumption and portfolio rules actually outperform (in a utility cost sense) both simple and bias-corrected empirical estimates of conditionally optimal policies.

*Keywords:* Return predictability; general equilibrium model; empirical experiments; optimal portfolio rules; relative utility cost.

## 1. Introduction

If monthly or quarterly excess returns on the market are predictable, then how should a rational investor adjust her consumption and portfolio choices in response to estimates of changes in expected returns?<sup>1</sup> Theoretically, time-varying expected returns introduce hedging demands and horizon effects into the investor's demand for the risky asset, and the qualitative nature of these effects has long been understood. A number of recent studies have attempted to quantify hedging demands and horizon effects by calibrating versions of the investor's problem to the amount of predictability found in US data.

These quantitative exercises have all been constructed in partial equilibrium. They start with a specific choice of the utility function of an investor, with a pre-determined investment horizon, and a description of the investment opportunity set, which consists of assumptions about the investor's initial wealth, non-asset income (if any), and the predictability of market returns in excess of the return on a risk-free asset. Given these basic building blocks, there are both frequentist and Bayesian approaches to evaluating the importance of predictability.

Frequentist studies, such as [Balduzzi and Lynch \(1999\)](#) and [Campbell and Viceira \(1999\)](#), are calibration exercises. They use point estimates of the parameters of the assumed data-generating process (DGP) and then produce point estimates of the optimal decision rules, including hedging demands and horizon effects. There is no explicit adjustment for uncertainty with respect to any of the features of the model. In contrast, Bayesian analyses, such as [Kandel and Stambaugh \(1996\)](#), [Barberis \(2000\)](#), or [Xia \(2001\)](#) explicitly incorporate some aspects of uncertainty into the computation of the investor's rules.<sup>2</sup>

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<sup>1</sup>See [Fama \(1991\)](#) for an early summary of a number of studies of predictability. See also [Campbell \*et al.\* \(1997\)](#) and [Cochrane \(1999\)](#).

<sup>2</sup>These studies address uncertainty about the value of a parameter(s) in a linear model. As a result, even this approach may suffer from a problem of mis-specification relative to the true data generating process. For example, in a model where the true relationship between dividend yields and excess returns is non-linear, these Bayesian models would be incorporating learning, but of the wrong kind.

In almost all cases, the DGP for excess returns is specified, exogenously, as a restricted version of a first-order vector autoregression (VAR) of (continuously compounded) excess returns and the (log of the) market dividend yield, driven by independent normally distributed shocks.<sup>3</sup> This choice has the virtue of simplicity, and it is generally regarded as an adequate approximation to the conditional distribution of returns and dividend yields. By necessity, these analyses are all based on the single sample path of realized US data.

We explore the quantitative impact of return predictability on a rational investor's optimal consumption and portfolio choices using a general equilibrium (GE) model as a laboratory for generating predictable returns. There are two advantages to using an explicit equilibrium model. First, the nature of return predictability and the form of the stochastic process for returns are, by construction, consistent with dynamic optimization of commonly used preferences. Second, our analysis is explicit about how portfolio rules relate to risk and are consistent with market clearing.<sup>4</sup>

The model is solved (numerically) and then simulated to produce asset returns and dividend yields. The dynamics of endogenously generated monthly excess returns and dividend yields are similar to those in the US data, at both short and long horizons. In particular, since there is little predictability in the monthly returns, the (conditional) Sharpe ratio in the simulated data, at the monthly frequency, does not exhibit substantial variation across states. As a result, the optimal, GE portfolio allocations of the time-separable, constant relative risk aversion investor varies by roughly 9% across extreme states.

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<sup>3</sup> [Avramov \(2002\)](#) considers multiple regressors using a Bayesian approach to evaluating model uncertainty, still within a linear model. [Brandt \(1999\)](#) estimates the optimal consumption and portfolio rules using a non-parametric estimate of excess return dynamics, which allows for nonlinearity in the conditional mean of excess returns. [Ait-Sahalia and Brandt \(2001\)](#) extend this approach to a more general consideration of time-varying first and second moments.

<sup>4</sup> As [Cochrane \(1999\)](#) notes, in the absence of an explicit understanding of the source of return predictability, the implications for portfolio rules are difficult to interpret: "If the (forecasted) premium is real, an equilibrium reward for holding risk, then the average investor knows about it but does not invest because the extra risk exactly counteracts the extra average return ... If the risk is irrational, then by the time you and I know about it, it's gone ... If the (forecasted premium) comes from a behavioral aversion to risk, it is just as inconsistent with widespread advice as if it were real. We cannot all be less behavioral than average, just as we cannot all be less exposed to a risk than average ..." (p. 72).

In examining the performance of quantitative estimates of portfolio rules applied to the simulated returns and dividend yields, we find that: (i) Quantitative rules constructed from simple ordinary least squares (OLS) point estimates of the DGP result in implied portfolio rules that are clearly biased and dramatically overstate the sensitivity of a rational investor's portfolio choice to the actual information in dividend yields. (ii) Since conditional means are difficult to measure precisely, the variance of the empirical optimal rules across multiple simulated histories of the model economy is very large. This suggests that calibration exercises based on point estimates from a single sample path can be very misleading. (iii) While a bias-correction strategy based on the standard linear VAR assumption for the DGP for returns improves upon the simple OLS-based empirical strategy, it still generates large utility costs for a significant proportion of the simulated model histories, when compared to the true optimal rules in the equilibrium model. (iv) In fact, in this setting, a simple unconditional (U) policy constructed from the point estimates of U mean and variance outperform the conditional rules, in a utility cost sense.

The decision rules of an investor with the same utility function and level of risk aversion as an agent in the model economy, constructed from estimates of a linear relationship between excess returns and dividend yields in the simulated data, show substantial variation over dividend yield states. For example, along a single sample path with the median level of predictability across the simulations of the economy, the portfolio weight in the risky asset ranges from  $-10\%$  to nearly  $350\%$ . Furthermore, under the empirical dynamics, the true optimal (GE) policy appears to have a substantial utility cost, relative to the empirical rules. A hypothetical econometrician attempting to quantify the optimal rules would conclude that the investor would be willing to forego between  $20\%$  and  $40\%$  of her initial wealth to switch *away* from the true GE rule towards what we know to be only an optimum under mis-specified empirical dynamics.

These single path results are broadly consistent with the existing literature. For example, [Balduzzi and Lynch \(1999\)](#) find that an investor facing no transactions costs, a coefficient of relative risk aversion equal to six, consumption at intermediate dates, and a 20-year horizon would be willing to pay nearly  $11\%$  of her initial wealth to have access to the information contained in a linear model that predicts risky (real) asset returns using the lagged market dividend yield. While  $11\%$  is at the low end of the range of utility cost calculations (based on wide differences in assumptions used in the literature), it still seems large, when compared to the level of

predictability in monthly returns.<sup>5</sup> Campbell and Viceira (1999), using different utility assumptions and an infinite-horizon investor, find that optimal investment varies from 60% to 140% (for expected log excess returns between 0% and 5%) and that suboptimal portfolio rules can imply large utility losses.<sup>6</sup>

The remainder of the paper is organized as follows: The model economy is defined in Sec. 2, and return dynamics are examined in Sec. 3. Section 4 compares a variety of consumption and investment policies with different assumptions about return dynamics. Utility cost calculations are contained in Sec. 5. The conclusions and suggestions for future research are in Sec. 6. Appendix A provides a proof of the proposition stated in Sec. 2.2, and Appendix B describes the algorithm used to solve the model and simulate the return data. Appendix C describes the optimal consumption/portfolio problem in a partial equilibrium setting, and it contains a description of the algorithm used to compute the optimal rules.

## 2. The Model Economy

### 2.1. Assumptions

A1: Decisions about the consumption of the single good and investment in the two marketed assets can be made continuously in time over an infinite horizon.

A2: There are two agents. Each agent is defined by the pair  $\{(U_i, W_{0i}), i = 1, 2\}$ , where  $U_i$  is the utility function and  $W_{0i}$  is the initial wealth allocation. Each agent is assumed to be representative of a large number of identical agents of the same type and, therefore, takes market prices as

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<sup>5</sup>Balduzzi and Lynch (1999) do not report the variation in the optimal risky asset holdings across dividend yield states for an investor who consumes at intermediate dates. However, the risky asset share for an otherwise identical terminal value of wealth maximizer varies between 30% and 84%, over most dividend yield states. Balduzzi and Lynch (1999) restrict the allocation to the risky asset to lie between 0% and 100% of the period wealth, as does Barberis (2000). This restriction may further reduce the utility cost as the conditional optimal policies may involve either short-selling or borrowing, as demonstrated by Brandt (1999).

<sup>6</sup>For example, for a time-separable, constant relative risk aversion investor with a coefficient of risk aversion equal to 10 who follows an optimal consumption policy but who ignores the information in predictable returns, the percentage loss in the value function is roughly 85%. This calculation is reported in Campbell and Viceira (2000), which corrects the calibration exercise in Campbell and Viceira (1999).

given. The utility functions for the two agent types are defined as follows:

- (i) Time-Separable (T-S) Agent: This agent type has a standard time-separable power utility function:

$$U_1(\{C_{1,t}\}_{t=0}^{\infty}) = E_0 \left[ \int_0^{\infty} \exp(-\beta t) \frac{-1}{1 - \gamma_1} dt \right], \quad (1)$$

where  $\beta$  is a constant time-discount parameter and  $C_{1,t}$  is the level of T-S Agent's time  $t$  consumption, and  $\gamma_1$  is the coefficient of relative risk aversion.

- (ii) Habit Agent: This agent type has the utility function:

$$U_2(\{C_{2,t}\}_{t=0}^{\infty}) = E_0 \left[ \int_0^{\infty} \exp(-\beta t) \frac{(C_{2,t} - \nu X_t)^{1-\gamma_2} - 1}{1-\gamma_2} dt \right], \quad (2)$$

where  $C_{2,t}$  is the level of the Habit Agent's time  $t$  consumption,  $X_t$  is an index of past aggregate consumption; i.e., the agent has external habit formation.  $\nu > 0$  is a constant that defines the intensity of the impact of past consumption on current utility. The habit index is defined as

$$X_t = X_0 \exp(-\phi t) + \phi \int_0^t \exp[\phi(s-t)] Y_s ds, \quad (3)$$

where  $Y_t$  is the exogenous aggregate endowment,  $\phi > 0$  defines the persistence of the impact of prior consumption on current utility.  $\gamma_2$  is a utility curvature parameter, but as shown in [Campbell and Cochrane \(1999\)](#), it is not equal to the coefficient of relative risk aversion. The Habit Agent has the same time-discount parameter as the T-S Agent.

A3: The natural logarithm of the exogenous aggregate endowment of the consumption good,  $\ln Y(t)$ , is:

$$d \ln Y_t = \mu dt + \sigma dB_t, \quad (4)$$

where  $B_t$  is a 1-dimensional Brownian motion.

[Chapman \(1998\)](#) demonstrates that in an endowment economy where the representative agent has internal habit formation preferences, the aggregate endowment process in Eq. (4) is incompatible with strictly positive state prices. A modification (used in the simulations reported below) to the geometric Brownian motion of Eq. (4) that preserves simple dynamics in the endowment growth rate and ensures positive

state prices is to set the diffusion coefficient of Eq. (4) to zero whenever

$$Y(t) = \pi\phi \int_0^t \exp(\phi(s-t)) Y(s) ds, \quad (5)$$

where  $\pi \geq \nu$ , which reflects the aggregate endowment back into the permissible region of the state space. In the simulations reported below, the constraint in Eq. (4) is never binding (see [Detemple and Zapatero \(1991\)](#) for a general treatment of the restrictions on preferences and endowments required to ensure strictly positive state prices in a representative agent internal habit model).

A4: There are two traded assets:

- (i) A default-risk free bond that is in zero net supply. Its price is denoted  $S^0(t)$ , and

$$\frac{dS_t^0}{S_t^0} = r_t dt,$$

where  $r_t$  is the instantaneous risk-free rate.

- (ii) A risky asset, with an ex-dividend price process  $S_t^1$ , that is a claim to future realizations of the aggregate endowment. The number of shares of the asset is normalized to one.

## 2.2. Equilibrium

The T-S Agent's optimization problem, at date  $t$ , is

$$\max_{\{C_{1,s}, \theta_{1,0,s}, \theta_{1,1,s}\}_{s=t}^{\infty}} E_t \left[ \int_t^{\infty} \exp(-\beta(s-t)) \frac{C_{1,s}^{1-\gamma_1} - 1}{1-\gamma_1} ds \right], \quad (6)$$

subject to the budget constraint

$$\begin{aligned} \theta_{1,0,t} S_t^0 + \theta_{1,1,t} S_t^1 + \int_0^t C_{1,u} du \\ = W_{01} + \int_0^t \theta_{1,0,u} dS_u^0 + \int_0^t \theta_{1,1,u} dS_u^1 + \int_0^t \theta_{1,1,u} dY_u, \end{aligned} \quad (7)$$

where  $C_{1,u} > 0$  is time  $u$  consumption,  $\theta_{1,0,u}(\theta_{1,1,u})$  is the number of units of the risk-free bond (risky asset) held on exit from time  $u$ , and  $Y_u$  is the aggregate dividend (endowment). The Bellman equation for this problem is standard. The solution will depend on the conditional expectation of future consumption choices, and the optimal portfolio and (current) consumption

choices will depend on two state variables: the current level of the endowment and the habit/endowment ratio.

The choice problem for the Habit Agent is more complicated, but it is standard in the literature on non-separable preferences:

$$\max_{\{C_{2,s}, \theta_{2,0,s}, \theta_{2,1,s}\}_{s=t}^{\infty}} E_t \left[ \int_t^{\infty} \exp(-\beta(s-t)) \frac{(C_{2,s} - \nu X_s)^{1-\gamma_2} - 1}{1-\gamma_2} ds \right], \quad (8)$$

subject to a budget constraint of the same form as Eq. (7) and the non-negativity condition  $C_{2,t} - \nu X_t > 0$ . Again, in equilibrium the stock and bond price dynamics will depend on the levels of two state variables so the optimal consumption and investment policy for this investor will be state dependent.

A *competitive equilibrium* is a set of asset price processes (generated as time-invariant functions of the model's state variables):

$$\left\{ S^0 \left( Y_t, \frac{X_t}{Y_t} \right), S^1 \left( Y_t, \frac{X_t}{Y_t} \right) \right\}_{t=0}^{\infty}$$

and consumption and portfolio decision rules (again, generated as time-invariant functions of the model's state variables):

$$\left\{ C_i \left( Y_t, \frac{X_t}{Y_t} \right), \theta_{i,j} \left( Y_t, \frac{X_t}{Y_t} \right) \right\}_{t=0}^{\infty}$$

for  $i = 1, 2$  and  $j = 0, 1$ , such that the following conditions are satisfied:

- (1) *Individual agent optimization*: The optimality conditions for the choice problems (6) and (8) are satisfied.
- (2) *Market Clearing*: The goods-market and the asset markets clearing conditions are satisfied; i.e.,

$$C_{1,t} + C_{2,t} = Y_t, \quad (9)$$

and

$$\theta_{1,0,t} + \theta_{2,0,t} = 0, \quad (10)$$

$$\theta_{1,1,t} + \theta_{2,1,t} = 1; \quad (11)$$

for all  $t$ .

The standard method of implementing a numerical solution to a complete markets competitive equilibrium is to solve the associated “social planner’s problem”, which specifies a linear combination (with constant coefficients) of the agents’ utility functions as the objective function and uses the goods



market clearing condition (Eq. (9)) as the budget constraint. The problem, in this case, is that the habit agent's objective function is not globally concave in consumption, and there is no general guarantee (for arbitrary parameter choices) that a solution to the planner's problem exists.

We address this issue in the following way: We choose the parameters of the utility function and the aggregate endowment process so that the conditions of Assumption 3.2 in [Detemple and Zapatero \(1991\)](#) hold at the aggregate endowment.<sup>7</sup> This implies that a rule that gives constant proportions of the endowment to each investor type is a feasible solution to the planner's problem with positive state-prices. The algorithm for maximizing the weighted-average utility then finds a feasible improvement. The numerical techniques used to compute the equilibrium allocations and prices are discussed in [Appendix B](#). In addition to giving the competitive consumption allocation as a function of the states, this solution method allows the calculation of asset price functions and the resulting return dynamics.

In examining the quantitative implications of the model economy, we will rely on the following result to simplify the analysis:

**Proposition.** *Under Assumptions A1–A4 and assuming that  $\gamma_1 = \gamma_2$ , dividend yields, optimal consumption shares, and equilibrium conditional and  $U$  expected asset returns vary only with the habit-endowment ratio,  $X_t/Y_t$ .*

**Proof.** See [Appendix A](#). □

**Remark.** It is not possible to establish an analogous result for portfolio shares, although it can be verified numerically that (under the assumptions of the proposition) that shares are also functions only of the habit endowment ratio.

### 2.3. Choosing the model parameters

There are no closed-form solutions for asset prices, consumption choices, and portfolio rules as functions of the model's state variables, which means that the equilibrium allocation must be computed numerically. The parameter values used to examine the model economy must balance two conflicting objectives: (1) The endogenous returns should be qualitatively similar to the actual US data, and (2) the risk aversion coefficients must be

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<sup>7</sup>As we noted earlier, that the geometric Brownian motion endowment is modified, if necessary, in order to ensure positive state-prices.

Table 1. Model parameters.

Name	Symbol	Value
Time Discount Rate	$\beta$	$\exp(-0.07)$
Agent 1's Utility Exponent Parameter	$\gamma_1$	8
Agent 2's Utility Exponent Parameter	$\gamma_2$	8
Habit Intensity	$\nu$	0.25
Habit Persistence	$\phi$	0.05
Social Planner's Weight on T-S Agent	$\lambda$	0.50
Endowment Growth	$\mu$	0.0172
Endowment Volatility	$\sigma$	0.06

consistent with the values used in the literature on measuring the effects of predictability.<sup>8</sup>

It is important to emphasize that the model economy is *not* intended as an explicit description of the full dynamics of US asset markets; i.e., we are not interested in attempting a formal analysis of the dimensions along which the model fits (or fails to fit) a wide range of asset pricing results. Instead, the model is intended as an *example* of an equilibrium framework in which expected excess returns are predictable and as a “mechanism” for generating simulated return data for subsequent analysis.

The parameterization of the model examined in the following sections is shown in Table 1. These values are chosen to match the basic dynamics of excess returns and dividend yields, at both short and long horizons, observed in the actual data. The endowment growth and volatility parameter values are higher than those consistent with aggregate consumption growth but lower than those implied by aggregate dividend growth rates.<sup>9</sup>

#### 2.4. *Some basic properties of the model economy*

The model was simulated 5,000 times, with each simulation consisting of 30 years of monthly observations. The results of these simulations are used to study the basic properties of the model economy. All of the optimal rules and endogenous asset returns are shown as functions of the habit/endowment ratio. In the following figures, the shaded histogram in the background of

<sup>8</sup>This rules out the parameterizations in [Campbell and Cochrane \(1999\)](#) and [Chan and Kogan \(2002\)](#). These models/parameterizations match asset returns more closely than the one we will consider further, but they use levels of risk aversion of 20 and higher.

<sup>9</sup>At parameter values for the endowment growth rate that are consistent with aggregate US consumption data, there will be an “equity premium puzzle” and a “risk-free rate puzzle”.

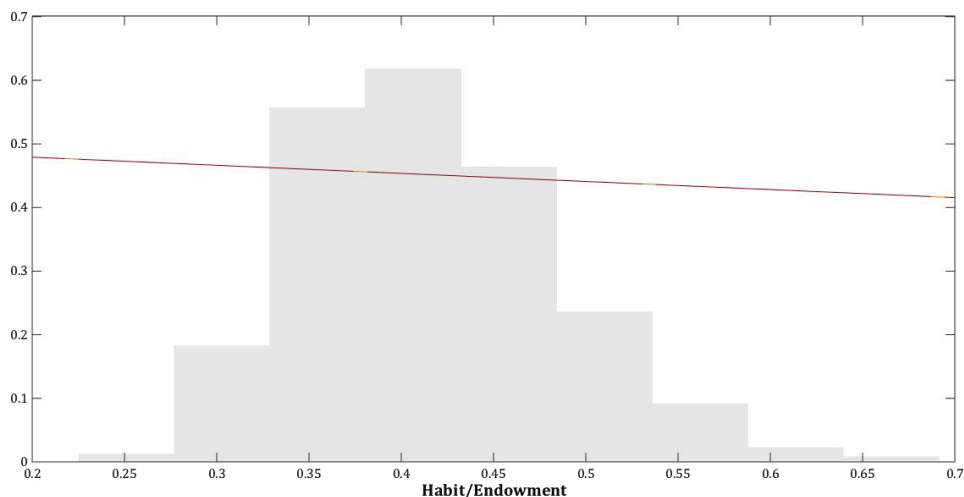


Fig. 1. The optimal consumption share of the total endowment for the time-separable investor as a function of the habit/endowment ratio (the true state variable). The shaded histogram shows the marginal density of the terminal value of the state variable across 5,000 simulations, each of length  $T = 360$  (months).

each panel shows the marginal density of the ratio of habit to endowment, across the simulated paths of the economy.<sup>10</sup>

The time-separable investor's optimal consumption share of the aggregate endowment is shown in Fig. 1. The consumption share is a linear, decreasing function of the ratio of habit to endowment, which seems intuitively reasonable, but this decrease is slight.<sup>11</sup> The optimal share fluctuates between 0.47 and 0.43 over the range of the ratio of habit to endowment. By construction, the optimal consumption share of the habit investor is one minus the share of the time-separable investor.

The optimal risky asset allocations for both agents, as a percentage of each agent type's wealth, are shown in Fig. 2. The top panel in the figure shows that the T-S investor is always issuing the risk-free asset to the habit investor and using the proceeds to invest in the risky asset. For different levels of the habit/endowment ratio, the T-S (Habit) investor holds between 104% (97%) and 113% (91%) of her wealth in the risky asset, over most of the support of the distribution of the habit/endowment ratio. The T-S investor's level of investment in the risky asset is nearly linear and increasing in the habit/endowment ratio. The optimal asset holdings of the habit investor must be

<sup>10</sup>This histogram is based on the 5,000 data points corresponding to the terminal value of the state variable.

<sup>11</sup>Linearity of the consumption share follows from Eq. (A.10) in Appendix A.

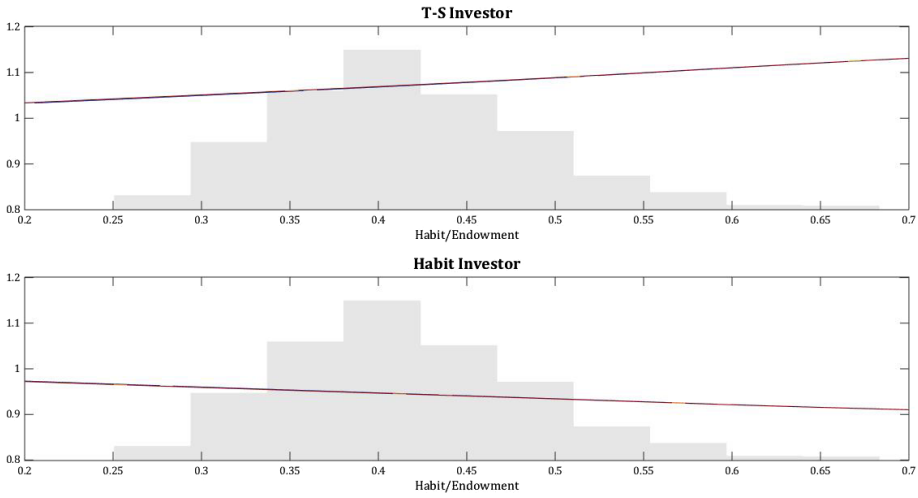


Fig. 2. The optimal risky asset portfolio allocation rules (as a percentage of each investor type’s wealth) as functions of the habit/endowment ratio (the true state variable). The shaded histogram shows the marginal density of the terminal value of the state variable across 5,000 simulations, each of length  $T = 360$  (months).

the opposite of the holdings of the T-S investor (in order to satisfy market clearing), and the lower panel of Fig. 2 shows that this is indeed the case.

Figures 3 and 4 show the dividend yield and the consumption/wealth ratio, respectively, as functions of the habit/endowment ratio. These figures document two important features of the model economy. At parameter

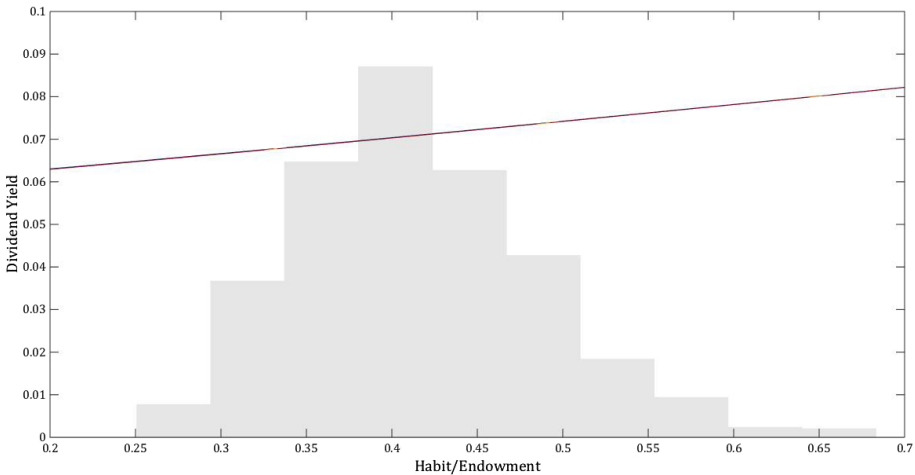


Fig. 3. The dividend yield as a function of the habit/endowment ratio (the true state variable). The shaded histogram shows the marginal density of the terminal value of the state variable across 5,000 simulations, each of length  $T = 360$  (months).

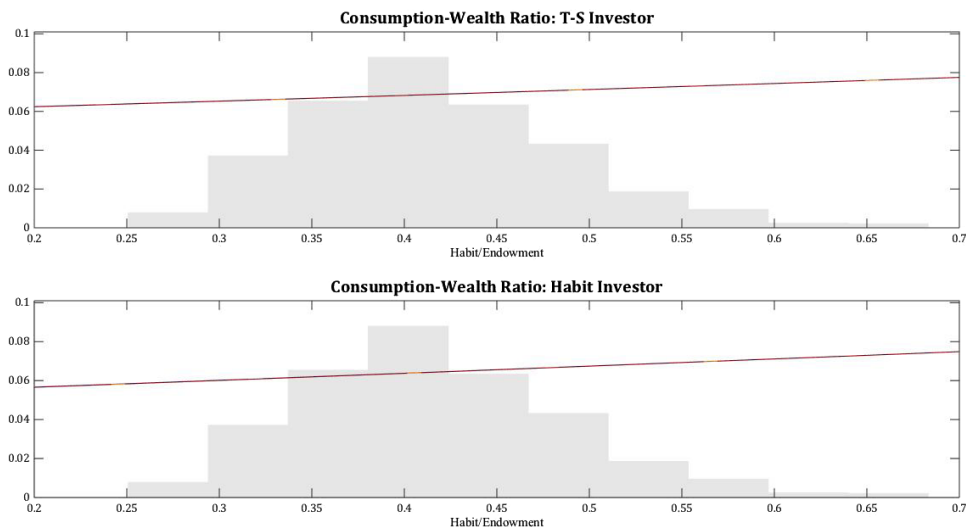


Fig. 4. The consumption/wealth ratio, for each investor type, as a function of the habit/endowment ratio (the true state variable). The shaded histogram shows the marginal density of the terminal value of the state variable across 5,000 simulations, each of length  $T = 360$  (months).

values that match the short-run predictability of excess returns found in the actual US data, there is little variation in true dividend yields and consumption/wealth ratios in the different states of the world. Perhaps more important, however, is the fact that dividend yields are a nearly linear function of the true state. While this fact is unobservable to an econometrician working with only returns and dividend yield data, the monotonicity in Fig. 3 implies that dividend yields are actually easily inverted to generate the habit/endowment ratio, and therefore they *should* be a good proxy for the true state.

The comparative insensitivity of GE rules to the true state of the economy can be explained in terms of the properties of the (conditional) Sharpe ratio

$$SR_t = \frac{E_t[R_{t+1}^1 - R_t^0]}{\sigma_{1t}}, \quad (12)$$

where  $R_{t+1}^1$  is the return on the risky asset from  $t$  to  $t + 1$ ,  $R_t^0$  is the return to the risk-free asset from  $t$  to  $t + 1$ , and  $\sigma_{1t}$  is the volatility of the risky asset. The Sharpe ratio, as a function of the habit/endowment ratio, is shown in Fig. 5. It is an increasing function, varying between 0.5 and 0.6.

The risk-free rate and the expected excess return on the risky asset, as functions of the habit/endowment ratio, are shown in Fig. 6. The model is

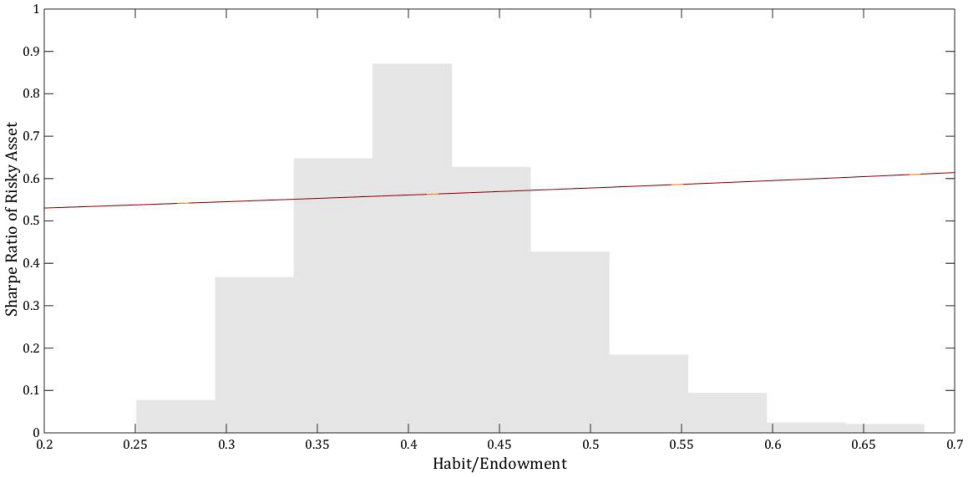


Fig. 5. The conditional Sharpe ratio as a function of the habit/endowment ratio (the true state variable). The shaded histogram shows the marginal density of the terminal value of the state variable across 5,000 simulations, each of length  $T = 360$  (months).

capable of generating both a time varying risk-free rate and a time-varying excess return, although since this is effectively a one-factor model, these changes will be perfectly correlated. Both risk-free and risky rates are increasing in the habit/endowment ratio, and the magnitudes of the variation in these quantities is comparable, both variables ranging between 4% and 5% per year.

As in [Campbell and Cochrane \(1999\)](#), the model generates state-dependent volatility in excess returns, and this dependence is shown in Fig. 7. The

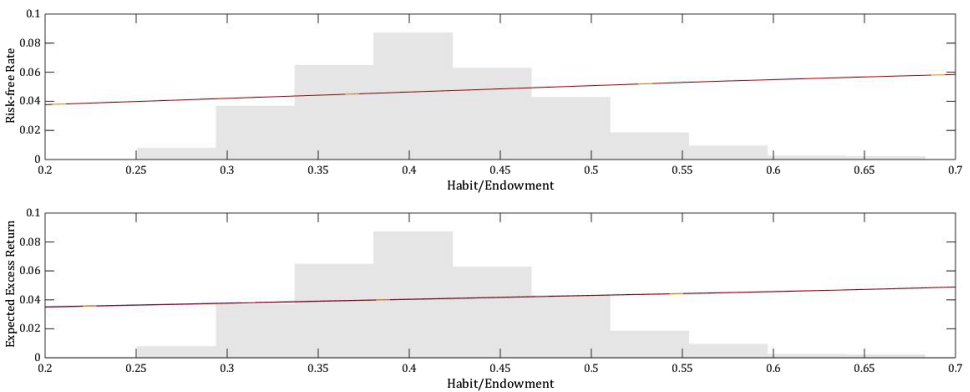


Fig. 6. The (real) risk-free rate and the expected excess return on the risky asset as functions of the habit/endowment ratio (the true state variable). The shaded histogram shows the marginal density of the terminal value of the state variable across 5,000 simulations, each of length  $T = 360$  (months).

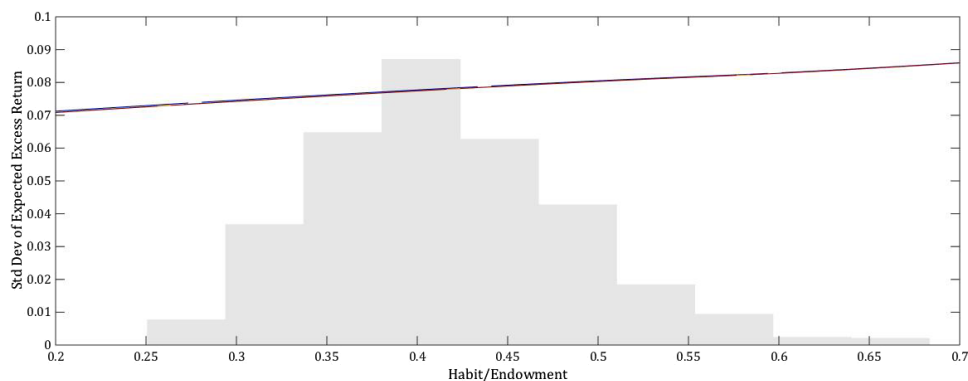


Fig. 7. The standard deviation of the excess return as a function of the habit/endowment ratio (the true state variable). The shaded histogram shows the marginal density of the terminal value of the state variable across 5,000 simulations, each of length  $T = 360$  (months).

standard deviation is an increasing function of the habit/endowment ratio, ranging from 7% per month, for low habit/endowment levels, to 8.5% per month for high habit/endowment levels. So, excess returns, in the model, are both higher and more volatile as the habit risk in the economy increases.

Summary statistics for model-generated excess returns and dividend yields, across 5,000 independent simulations of the model economy are shown in Table 2. The data simulated from the model have the following qualitative

Table 2. Summary statistics for simulated excess returns and dividend yields.

	Mean	Std. Dev.	Autocorrelation (One-Month)
<b>Excess Return</b>			
Mean	0.0426	0.0737	-0.0030
Median	0.0424	0.0737	-0.0048
25th Percentile	0.0342	0.0717	-0.0386
75th Percentile	0.0508	0.0756	0.0343
<b>Dividend Yield</b>			
Mean	0.0713	0.0017	0.9806
Median	0.0711	0.0016	0.9834
25th Percentile	0.0699	0.0013	0.9752
75th Percentile	0.0726	0.0021	0.9894

*Notes:* Summary statistics for the continuously compounded excess returns and log dividend yields from 5,000 simulated sample paths of the model economy. Means are annualized by multiplying monthly values by 12 and standard deviations are annualized by multiplying monthly values by  $\sqrt{12}$ .

Table 3. Summary statistics for US excess returns and dividend yields: 1961:01 to 1998:12.

	Excess Returns	Dividend Yields
Mean	0.0654	0.0334
Median	0.0964	0.0320
Standard Deviation	0.1520	0.0025
25th Percentile	-0.2399	0.0284
75th Percentile	0.4107	0.0389
1st-Order Autocorrelation	0.0647	0.9842

*Notes:* Excess returns are the continuously compounded monthly return to the CRSP value-weighted index between  $t$  and  $t + 1$  in excess of the continuously-compounded one-month Treasury bill yield at time  $t$ , where the one-month yield comes from Ibbotson Associates. Means are annualized by multiplying monthly values by 12 and standard deviations are annualized by multiplying monthly values by  $\sqrt{12}$ . Log dividend yields are constructed, as in Fama and French (1988), from the with- and without-dividend return to the CRSP value-weighted portfolio.

properties: (i) the stationary distributions for excess returns and dividend yields are approximately symmetric; (ii) the average risk premium is 4.26% per year with a standard deviation of 7.37% per year; (iii) the average value of the dividend yield is 7.13% per year with a standard deviation of 0.17% per year; and (iv) excess returns are virtually uncorrelated and dividend yields are highly autocorrelated.

For comparison purposes, Table 3 contains summary statistics on the continuously-compounded monthly excess returns to the CRSP value-weighted portfolio and the log dividend yield (for the CRSP value-weighted portfolio) from 1961 to 1998. The model results for excess returns are broadly consistent with the data, although the empirical distribution of actual excess returns is not symmetric. Dividend yields in the model are as persistent as actual dividend yields, although their average value is more than twice the level of the actual data. The volatility of model generated excess returns is slightly less than half of the volatility of actual excess returns. Model generated dividend yields are about 70% as volatile (on average) as actual dividend yields. These facts are important because they imply, if anything, that our model tends to overstate the predictability of expected excess returns. The overall conclusion that we draw from a comparison of Tables 2 and 3 is that the parameterization of the model used to simulate the return and dividend yield data is adequate for the task at hand.



### 3. Return Predictability

#### 3.1. Empirical dynamics in the US data

There is an extensive literature that documents the predictability of different equity portfolios over different time periods and different holding period horizons. Table 4 contains information from a number of representative studies describing the results from linear regressions of monthly market returns on lagged values of annual market dividend yields.

At monthly or quarterly horizons, a few robust facts emerge from the OLS regressions in Table 4: (i) Higher dividend yields are positively related to future returns; (ii) About half of the slope coefficients in the regressions are not significantly different from zero;<sup>12</sup> (iii) the  $R^2$  statistics in these regressions, while larger at the quarterly than the monthly horizon, never exceed 5.2% in any case; and (iv) while the point estimates of the slope coefficient vary, the general results are robust to whether or not returns are *ex post* real

Table 4. A summary of return predictability based on dividend yield regressions.

Authors (Year)	Data Characteristics			Regression Characteristics		
	Period	Frequency	Other	Slope	Slope- $t$	$R^2$
Fama–French (1988)	1941–1986	$M$	$r$	0.280	1.830	0.000
Fama–French (1988)	1941–1986	$Q$	$r$	1.260	2.480	0.020
Campbell <i>et al.</i> (1997)	1952–1994	$M$	$r$	0.027	3.118	0.018
Campbell <i>et al.</i> (1997)	1952–1994	$Q$	$r$	0.080	3.152	0.049
Balduzzi–Lynch (1999)	1927–1991	$M$	$r$ , VAR	0.304	<i>n.r.</i>	0.004
Campbell–Viceira (1999)	1947–1995	$Q$	$x$ , VAR	0.069	3.286	0.052
Barberis (2000)	1927–1995	$M$	$r$	0.258	1.806	0.004

*Notes:* The dependent variable in the regression is the real return to a value-weighted market proxy, and the independent variable is annual dividend yield on the market (except for Campbell *et al.* (1997), Balduzzi and Lynch (1999), and Campbell and Viceira (1999) who use the log dividend yield). In the “Data frequency” column,  $M(Q)$  refers to monthly (quarterly) observations. In the “Data Other” column, “ $r$ ” means real returns were used, and “ $x$ ” means excess returns were used; and “VAR” means that the return versus dividend yield relationship was estimated as part of a larger VAR. “Slope” in the regression characteristics section refers to the slope coefficient in the linear regression, while “Slope- $t$ ” is the  $t$ -statistic on the slope coefficient.  $R^2$  is the percentage of the variation in returns explained by the regression model. “*n.r.*” means that this statistic was not reported.

<sup>12</sup>In all of the studies cited in Table 4, the  $t$ -statistics in the regression are calculated using standard errors that are robust to heteroskedasticity and serial correlation in the regression residuals.

returns or returns in excess of a short rate and whether or not they are from a single-equation model or a (restricted) bivariate VAR.

There are two well-known problems associated with interpreting the statistical significance of the regressions summarized in Table 4. First, dividend yields are highly persistent series. For example, Campbell and Viceira (2000) report that the coefficient of log dividend yields on lagged log dividend yields in their bivariate VAR is 0.957, and Cochrane (2001) reports a value of 0.97 for annual log dividend yield autoregressions. Using monthly post-War data through 1999, the autocorrelation of log dividend yields is closer to 0.99. This issue is important because *near unit root* behavior in a time series regression can have an important impact on the accuracy of conventional asymptotic approximations in finite-sample; i.e., it is very difficult to interpret a *t*-statistic of 3.152 as actually coming from a Student *t* sampling distribution.<sup>13</sup>

The second problem is that log dividend yields are pre-determined and not exogenous regressors. Stambaugh (1999) notes that, in the regression of returns on lagged variables related to end of period asset prices, regression disturbances are generally correlated with lagged and future values of the regressor. The resulting OLS parameter estimates, while consistent, are biased in finite-sample, and the extent of this bias is related to the persistence of the regressor. In an example of the return versus dividend yield regression calibrated to US data from 1952 to 1996, Stambaugh (1999) finds that the upward bias is nearly as large as the OLS estimate itself and the sampling distribution deviates substantially from the normal distribution.

These results suggest that the statistical significance of the regression of monthly and quarterly market returns on lagged market dividend yields is difficult to interpret. In particular, the point estimates of the parameters may deviate substantially from the range of values reported in Table 4, and calibrating a model to precisely reproduce these estimated coefficients may be misleading. Nonetheless, as will be demonstrated below, the model delivers regression results that are broadly consistent with the point estimates reported in Table 4.

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<sup>13</sup>Goetzmann and Jorion (1993) and Nelson and Kim (1993) are early Monte Carlo studies demonstrating finite-sample biases in long-horizon return versus dividend yield regressions. Valkanov (2003) is a comprehensive recent theoretical treatment of near unit root behavior in long-horizon predictability regressions, including those that use dividend yields to forecast excess market returns.

### 3.2. Empirical dynamics in the model-generated data

Given a time series of model-generated excess returns and dividend yields, we estimate the simple (restricted) VAR for excess returns and dividend yields commonly used in the literature:

$$\begin{aligned} R_{t+1}^1 - R_t^0 &= \alpha + \beta \log(D_t/S_t^1) + \varepsilon_{1t+1}, \\ \log(D_{t+1}/S_{t+1}^1) &= \mu + \rho \log(D_t/S_t^1) + \varepsilon_{2t+1}, \end{aligned} \quad (13)$$

where  $R_{t+1}^1$  is the continuously compounded return on the risky asset from  $t$  to  $t+1$ ,  $R_t^0$  is the continuously compounded return to the risk-free asset from  $t$  to  $t+1$ ,  $D_t$  is the sum of the endowment from  $t-11$  through  $t$ , and  $S_t^1$  is the time  $t$  price of the risky asset. The parameters in (13) can be estimated (and are typically estimated) by applying OLS to each equation, individually. In any sample of simulated data, the OLS estimate of  $\beta$  will suffer from the finite-sample bias examined in [Stambaugh \(1999\)](#), but since this issue has not been corrected in the empirical literature, we will (initially) follow that convention here.

Histograms of the point estimate of the slope coefficient and the  $R^2$  statistic of the excess return regression in Eq. (13), for 5,000 simulated sample paths of 30 years worth of observations from the model economy, are shown in Fig. 8. Consistent with the evidence in Table 4, the  $R^2$  of the regression is typically less than 1%. The density of the slope coefficient in the simulated returns versus dividend yield regressions is skewed to the right and centered

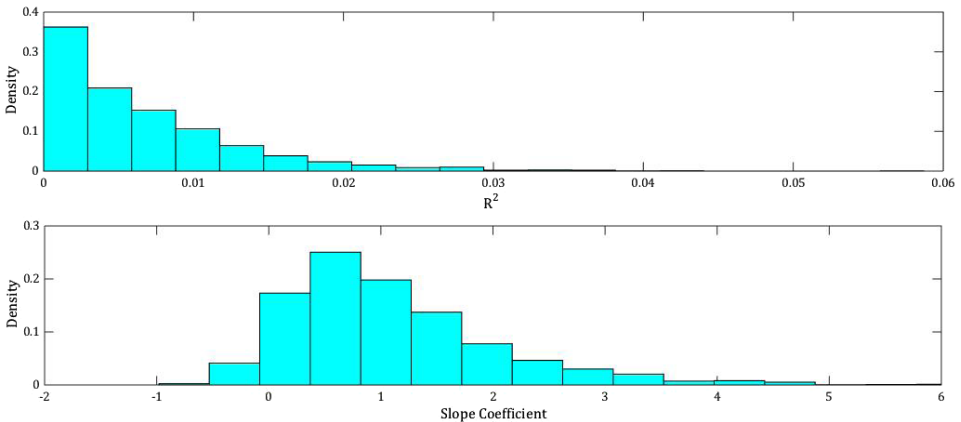


Fig. 8. The empirical densities of the  $R^2$  statistic and the slope coefficient in the regression for one-month excess returns on lagged dividend yield, across 5,000 simulations, each of length  $T = 360$  (months).

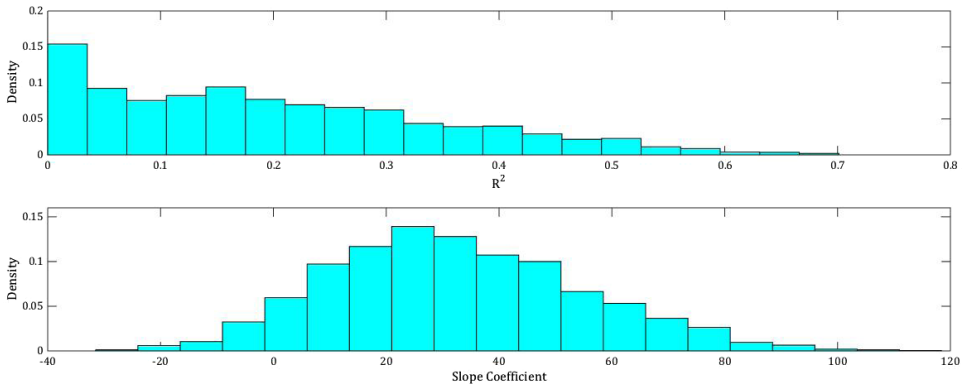


Fig. 9. The empirical densities of the  $R^2$  statistic and the slope coefficient in the regression for 3-year excess returns on lagged dividend yield, across 5,000 simulations, each of length  $T = 360$  (months).

near 1.0, although there is considerable spread in this distribution. These values are consistent with their empirical counterparts reported in Table 4.

The predictability of long-horizon returns is examined in Fig. 9, which is the long-horizon (three-year) analog of Fig. 8. The explanatory power of the long-horizon excess return regressions is substantially higher than those of the one-month horizon regressions. The distribution of the  $R^2$  statistic, across the 5,000 simulated histories of the model, places approximately 25% of the probability mass between 0.20 and 0.40. The slope coefficients are also larger, with a mean value of approximately 30%. These results are consistent with the long-horizon regressions results reported in Campbell *et al.* (1997) and Cochrane (2001).

### 3.3. *Serial correlation and normality of the model generated residuals*

In constructing empirical estimates of optimal policies, it is important to verify whether or not the measured residuals to the simulated data are consistent with being independent draws from a normal distribution. If these conditions are consistent with the simulated data, then the bias-correction suggested in Stambaugh (1999) is feasible and appropriate. In order to address this question, we first apply a standard serial correlation test to the each of the  $T = 360$  samples of model residuals. The results are reported in Table 5.

The Box–Pierce test statistic, which is constructed as a weighted-average of a specified number of the squared residual sample autocorrelation

Table 5. Serial correlation and normality tests for VAR residuals.

	Nominal Rejection Rate (%)						
	5	10	25	50	75	90	95
Panel A: Rejection Rates for the Box–Pierce Test (w/12 Lags)							
Simulated Residuals							
Excess Returns	4.32	9.48	23.64	48.84	74.20	89.76	95.44
Dividend Yield	6.24	12.16	27.52	53.56	77.08	92.00	96.00
Panel B: Rejection Rates for the Jarque–Bera Test							
Simulated Residuals							
Excess Returns	4.84	10.12	26.72	54.80	79.88	92.00	95.80
Dividend Yield	8.60	12.72	27.00	51.28	76.98	90.20	94.88

*Notes:* The Box–Pierce statistic is constructed using a weighted sum of 12 lagged, squared autocorrelations. Under the null hypothesis of no serial correlation, it has an asymptotic  $\chi^2_{12}$ -distribution. The empirical rejection rates for 5,000 simulated time series of length  $T = 360$  are compared to the rejection rates under the asymptotic distribution. The Jarque–Bera test evaluates the null hypothesis that a set of observations comes from a normal distribution with unknown mean and variance against the alternative that the sample does not come from a normal distribution using the sample skewness and kurtosis.

coefficients has an asymptotic chi-square distribution, with degrees of freedom equal to the number of squared autocorrelations used in constructing the statistic. We compute the value of this statistic (using twelve lags of each residual series) for each of the 5,000 sample paths. The critical values based on this large sample is then compared to the critical values of the asymptotic distribution, under the null of no serial correlation. The results in Panel A of Table 5 suggest that the residual series are very close to being uncorrelated. The actual rejection rates of the test statistics correspond very closely to what we would expect by random chance under the asymptotic distribution.

Given that the residuals are uncorrelated, we can also test for normality. The Jarque–Bera test uses the sample skewness and kurtosis to test the null hypothesis that the data is drawn from a normal distribution. The results in Panel B confirm that normality of the VAR residuals is not inconsistent with the simulated data. In summary, a hypothetical econometrician faced with the simulated data would reasonably conclude that the data conform quite nicely to the standard assumption of a restricted first-order VAR driven by normally distributed innovations. A simple empirical (SE) bias-correction seems quite reasonable, under these conditions.

#### 4. Comparing Optimal Policies

Since the empirical literature, by and large, focuses on a time-separable, constant relative risk aversion investor, we will as well. Table 6 lists a number of possible sources of differences between GE rules, and the empirical rules. Perhaps the most important issue to examine is that the true relationship between the risk premium and the state variable is known in the model, but the empirical model specifies a linear relationship between return and dividend yield and the slope coefficient is biased in finite sample.

In this and the following section, we will examine the relative importance of these specification issues for consumption policies and portfolio rules, as well as for the interpretation of utility cost calculations. Appendix C describes the general form of the partial equilibrium consumption/portfolio problem and the numerical methods that we use to solve it. We will consider the following consumption/portfolio rules:

- **SE:** The optimal policies for a CRRA agent using the linear, empirical dynamics, based on the simulated data from GE model, without any attempt at a bias-correction for the slope coefficient. This corresponds to the standard practice in the literature on the quantitative significance of portfolio choice under return predictability.
- **GE:** The optimal consumption and portfolio choices from GE model. Given that  $\alpha^* = f(X/Y)$  and  $(X/Y) = h(D/S^1)$  and that  $f$  and  $h$  are monotonic functions, we can write the optimal portfolio rule as the composite function  $\alpha^* = f \circ h(D/S^1)$ .

The difference between SE and GE, across values of the dividend yield, is the quantity of interest; i.e., how misleading a picture of GE rule do we see in the empirical estimates?

Table 6. Differences between the true and empirical models.

True Model	(Standard) Empirical Model
1. $(R_{t+1}^1 - R_t^0)$ versus $(D_t/S_t^1)$ is known	$\hat{\beta}$ from $(R_{t+1}^1 - R_t^0) = \alpha + \beta(D_t/S_t^1) + \varepsilon_{t+1}$ is biased
2. $var_t(R_{t+1}^1 - R_t^0) = f(D_t/S_t^1)$	Constant Volatility
3. $(D_t/S_t^1)$ is highly persistent	$\hat{\rho}$ from $(D_{t+1}/S_{t+1}^1) = \gamma + \rho(D_t/S_t^1) + \eta_{t+1}$ is biased downward
4. $\infty$ -horizon	Finite horizon
5. $R^0$ is stochastic.	$R^0$ is constant
6. Continuous time	Discrete time

Notes:  $(R_{t+1}^1 - R_t^0)$  is the excess return on the risky asset at date  $t + 1$ .  $(D_t/S_t^1)$  is the dividend yield at time  $t$ .  $R^0$  is the return to the risk-free asset.

- **Empirical Bias-Corrected (EBC):** This rule is computed by solving the CRRA investor's portfolio problem using the slope coefficient adjusted by the normality based correction in [Stambaugh \(1999\)](#). This bias correction uses information about the slope coefficient in the dividend yield autoregression and the covariance matrix of the residuals in the VAR in (13).<sup>14</sup>
- **Model Bias-Corrected Empirical (MBC):** This rule is computed by solving the CRRA investor's portfolio problem using the true functional form for the dependence of the conditional mean of excess returns on dividend yields. Although this rule could not be known in practice, it provides important information about the proportion of the difference between SE and GE that can be attributed to the bias problem. It also provides important information on the accuracy of the simulations.
- **U Policy:** This is the rule that would emerge from a Merton–Samuelson style analysis based on unconditional moments.

#### 4.1. *Single path analysis*

The optimal policies, under the various specifications defined in the previous subsection, are shown in Fig. 10. The policies, SE, EBC, MBC, and U, are constructed, as described above, using the dividend yield as the conditioning variable. The dividend yield is assumed to follow a first-order autoregressive process with Gaussian innovations, and the innovations to the excess return are also assumed to be Gaussian with constant variance. This is typically done in the literature, and as the last section indicated, it is a reasonable approximation to the model generated data.

<sup>14</sup>The precise form of the correction follows from the following approximation to the OLS,  $\hat{\beta}$ , bias:

$$E[\hat{\beta} - \beta] = -\frac{\sigma_{uv}}{\sigma_v^2},$$

where  $\sigma_{uv}$  is the contemporaneous correlation between the disturbance to the predictive regression and the dividend yield autoregression,  $\sigma_v^2$  is the variance of the disturbance to the dividend yield autoregression,  $T$  is the sample size, and  $\rho$  is the slope coefficient in the dividend yield autoregression (see [Stambaugh, 1999](#), Sec. 2). Evaluating the bias requires specifying values for  $\sigma_{uv}$ ,  $\sigma_v^2$ , and  $\rho$ . While the first two parameters can be approximated by OLS, this is clearly problematic in estimating  $\rho$ , given the known downward bias in  $\hat{\rho}$  in an autoregression with a root near one. Instead, we solve for the implied  $\rho$  using

$$E[\hat{\rho} - \rho] = -\frac{1 + 3\rho}{T}.$$

In any case, [Stambaugh \(1999\)](#) argues that this effect is small.

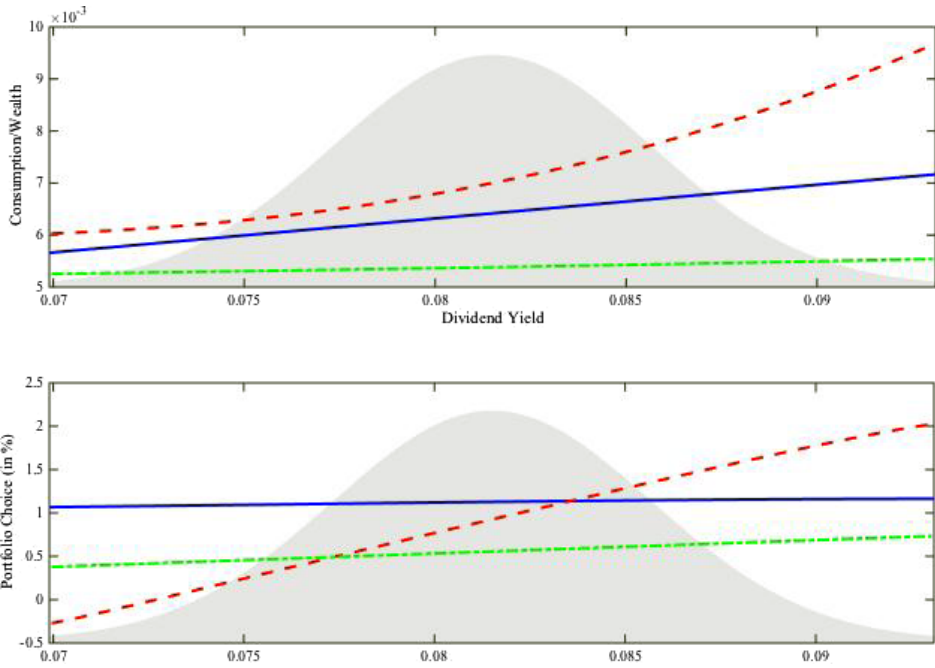


Fig. 10. Optimal consumption/wealth ratios and risky asset portfolio allocations, as functions of the dividend yield. The rules are computed for a sample path with the median slope coefficient in the simple regression of excess returns on dividend yields. The planning horizon is 50 years. SE is the simple empirical rule, GE is the general equilibrium rule, U is the unconditional (Merton–Samuelson) rule, EBC is the empirical bias-corrected rule, and MBC is the bias-corrected rule using the true form of the conditional expected excess return, as a function of the dividend yield.

The results in Fig. 10 are for a sample path that has the median slope coefficient, from the distribution of the 5,000 sample paths, in the linear regression of excess returns on dividend yields. In particular, the T-S investor is assumed to observe 30 years of monthly observations of excess returns and dividend yields, which are used to estimate the parameters required to implement the specific consumption/portfolio rule. The policy and its implied utility are then calculated, using a 50-year horizon, which is chosen to reduce the discrepancy between the empirical and GE policies that can be attributed to the infinite-horizon in the model economy.<sup>15</sup> For the SE, EBC, MBC and U investment and consumption policies, only the choices in the initial period are shown. The GE policy is time-invariant and is also plotted here, for comparison purposes, as a function of the dividend yield.

<sup>15</sup>We have also examined other paths at both 10- and 20-year horizons and results are qualitatively similar. These results are available upon request.



There are dramatic differences between the empirical and GE policies, for both consumption and portfolio choice. The top panel of Fig. 10 describes the consumption/wealth ratio as a function of dividend yield. The optimal policy SE is very different from GE rule. An agent with this perception of return behavior consumes at a much higher rate across all states. This follows because of the wealth effect due to increased expected return and the fact that the risky asset is a normal good. In addition to a higher level, the implied variability in consumption under SE is also much higher across the different dividend yield states.

The bottom panel of Fig. 10 shows that the SE portfolio rules are substantially more state dependent than GE policies. For example, implementing the SE policy would imply a wealth share devoted to the risky asset that fluctuates from approximately  $-10\%$  for a dividend yield realization in the extreme left tail of the cross-sectional distribution of dividend yields to approximately  $350\%$  in the extreme right tail. The variability of the SE rule across the interquartile range of dividend yields reported in Table 2 is also quite large, ranging from  $10\%$  to more than  $100\%$ . The results in Fig. 10 are similar to the empirical policies corresponding to the calibration of predictability in [Balduzzi and Lynch \(1999\)](#) and [Campbell and Viceira \(1999\)](#).

To see the effect of this bias on consumption and portfolio choices, we examine conditional policies, EBC and MBC. The top panel of Fig. 10 shows that, when compared to the SE consumption policy, the consumption rules EBC and MBC are both lower in level and less variable. In the bottom panel of Fig. 10, by explicitly correcting the biases in SE calibration, we find that the portfolio choices in EBC and MBC become much less variable. All of these portfolio rules suggest risky asset holdings near  $100\%$  of the investor's allocation. The MBC portfolio rule is close to the true optimal (along this one path), suggesting that the numerical approximations to GE and partial equilibrium policies are consistent, and that the associated approximation errors are not too large. The EBC rule — which in contrast to the MBC rule actually *can* be implemented — is virtually indistinguishable from the MBC rule, on this path.

In conclusion, we emphasize that all of these policies, excluding GE, are obtained using the statistics computed along one realized sample path: the one that has the median slope in the return predictability regression. The alternative rules can be computed and examined along a collection of realizations of the model economy. This exercise provides important information about the variability of the estimated rules, and their associated utility costs, about the true optimal rules.

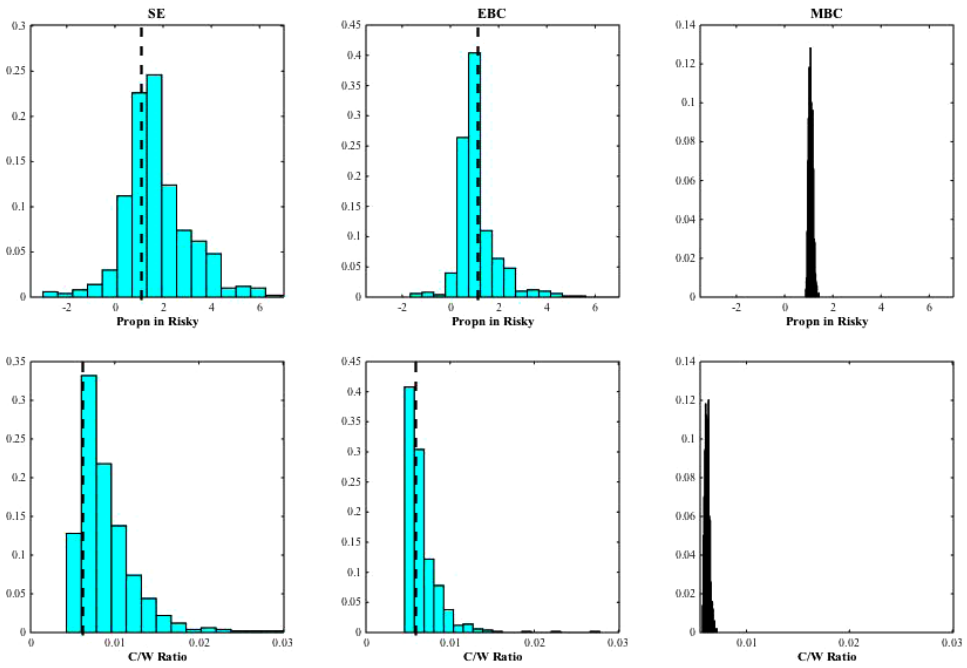


Fig. 11. Distribution of SE, EBC, and MBC optimal risky asset portfolio shares and consumption/wealth ratios evaluated at the median of the sampling distribution for dividend yields across 500 simulated sample paths, using a planning horizon of 50 years. The vertical dashed-line in the first two columns of graphs shows the horizon of the MBC density.

#### 4.2. Multiple path analysis

Are the results in Fig. 10 representative of a typical outcome from a simulated history of the model economy in Sec. 2? In order to answer this question, we simulate 500 independent histories and compute the SE, EBC, and MBC rules along each path.<sup>16</sup> Figures 11 and 12 summarize the information generated by this experiment.

The top row of the graphs in Fig. 11 shows the distribution, at the median of the cross-sectional sampling distribution for dividend yields, of the proportion of wealth invested in the risky asset under the SE, EBC, and MBC alternatives for measuring excess return predictability. Given the results in Figs. 2 and 6, the true optimal portfolio allocation should be approximately 105% of the T-S agent's wealth. The actual distribution of proportions, under the SE rule, ranges from  $-200\%$  to in excess of  $600\%$ . The bulk of the distribution of proportions is between  $0\%$  and  $400\%$ , with most of the mass of the estimates well above  $105\%$ . The overall impression, from the top left

<sup>16</sup>Our analysis was limited to 500 paths due to computational intensity.

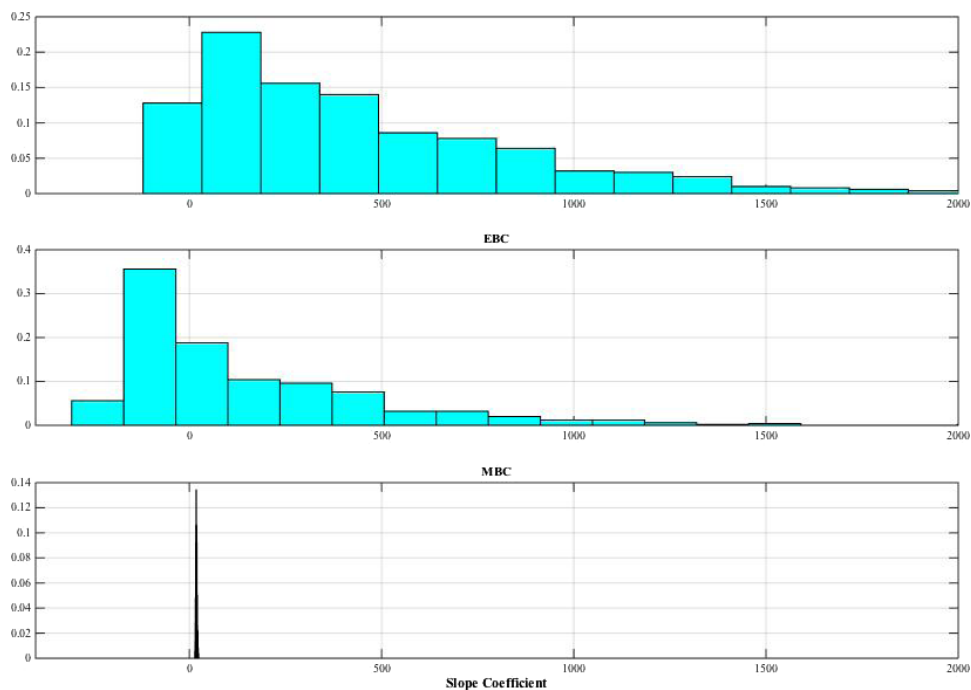


Fig. 12. Distribution of the slope of the optimal portfolio rule for SE, EBC, and MBC estimates for 500 sample paths, with a planning horizon of 50 years.

histogram in Fig. 11, is that the SE rule generates a wide dispersion in the estimates of the optimal rule, at the median of the dividend yield distribution.

The EBC rule, constructed using the bias-correction based on the normal distribution, as in [Stambaugh \(1999\)](#), and shown in the middle column of Fig. 11, reduces the dispersion of the estimated rule (at the median dividend yield). This reduction in cross-sectional variance occurs because, along each path, the bias-correction also depends on the parameters of the residual covariance matrix. It also centers the distribution of estimated portfolio shares closer to 1. However, it still shows substantial variation in the computed rules, with the bulk of the distribution falling between 0% and 300%.<sup>17</sup>

The MBC rule, which is not feasible in practice, is clearly the most precise of the three rules. The results for this rule are shown in the top right panel of Fig. 11. The histogram of estimated portfolio proportions has its center very

<sup>17</sup>In an experiment not reported here, when the data generating process is a linear VAR, the *OLS* slopes across simulated paths are positively biased. In this case, the empirical bias correction, does reduce the bias in the slopes, but the variance of the distribution of the corrected slopes remains largely unchanged. This suggests that the conditional volatility documented in Fig. 7 has an important effect on the empirical rules generated from the simulated data.

close to the true optimal value.<sup>18</sup> More important, however, is that the estimated values are very precise, in the sense that the sampling distribution has a very small variance (when compared to either of the empirical rules). This result says two things: (i) getting the conditional expectation “right” is very important, and (ii) the simulation procedure is sufficiently accurate that it is meaningful to interpret the results of the simulations.

The bottom row of histograms in Fig. 11 shows the estimates, at the median dividend yield value, of the computed optimal consumption/wealth ratio, according to the SE, EBC, and MBC rules. It demonstrates that the relative features for the optimal portfolio proportion (in the top row of the figure) are also found in the optimal consumption rule. In particular, the SE rule generates the most dispersed and biased measures of the optimal consumption/wealth ratio, the EBC is an improved but still flawed estimate, and the MBC rule (while not empirically feasible) is quite accurate and precise.

Figure 11 only measures the estimated portfolio rule at a single point in the stationary distribution of dividend yields. In order to ensure that these results are representative of the entire estimated rule, we present the distribution of the slope parameters of the optimal portfolio share rule (as a function of dividend yields), across the 500 sample paths, in Fig. 12. Since these optimal rules are very nearly linear, the distribution of the slopes provides an accurate measure of the variability of the rules across the different paths.

The results in Fig. 12 are consistent with those in Fig. 11.<sup>19</sup> The distribution of the slope parameters from the SE rule are both very dispersed and biased upward, given that the true slope is close to zero. The EBC rule, as in Fig. 11, is an improvement over SE, but it is still quite variable, with a large proportion of estimated paths containing slopes in excess of 100. The MBC rule is, again, both accurate and precise, across alternative histories of the model.

In summary, Figs. 11 and 12 confirm that the SE rule is generally quite inaccurate, and they show that the SE rule is very widely dispersed for different realized histories from the model economy. *This suggests, quite strongly, that rules based on uncorrected OLS estimates of excess return predictability are generally not reliable estimates of the true optimal rules.* The multiple path results also show that the EBC rule is an improvement on the SE rule, but not as strong an improvement as suggested by Fig. 10. In the

<sup>18</sup>The mean of the MBC distribution is 115%.

<sup>19</sup>Negatively sloped optimal portfolio rules correspond to the few (permissible but pathological) cases in which the point estimate of the slope coefficient in the dividend yield autoregression is greater than one.

single path analyzed in Fig. 10, the *EBC* rule appears to be essentially as good an approximation to the true rule as the *MBC* rule, which uses accurate information about the DGP that is unavailable to the empiricist. The multiple path results show that — while this outcome is possible — it is not representative of the vast majority of simulated histories of the model economy.

One point that is not clear from Fig. 10 (or Figs. 11 or 12 for that matter) is how we should judge the distance between the alternative rules. In Fig. 10, is *EBC* actually “close” to *GE*, or would the T-S agent perceive them as very different? The natural metric for examining the distance between paths is to consider the utility costs of switching between alternative investment rules, and this is what we turn to next.

## 5. The Utility Costs of Alternative Strategies

One intuitive — and commonly used — measure of the utility cost of switching between alternative rules asks the following question: How much wealth would an agent be willing to give up in order to incorporate information about predictability into his or her choices? In this section, we examine the utility costs of pursuing a variety of policies in three alternative settings.

We first calculate utility costs from the point of view of the empiricist who uses a simple *OLS* estimate of the predictability of excess returns. Next, we consider the utility cost calculations of an empiricist who knows about the finite-sample bias in the *OLS* estimates and corrects predictability measures using the normality based correction in [Stambaugh \(1999\)](#); i.e., an investor using the *EBC* rule. Finally, we calculate utility costs using the equilibrium relationships among the state variables and relevant portfolio choice variables.

The utility cost of a policy  $\pi_\bullet$  in setting  $S$  is defined as the portion of wealth an agent would have to forgo in order to be indifferent between the policy  $\pi_\bullet$  and the strategy  $\pi_S$ , which is optimal in the specified setting.<sup>20</sup> The utility cost is  $W$  in

$$U_S(\$1, \pi_\bullet | Z_t) = U_S(\$1 - W, \pi_S | Z_t), \quad (14)$$

where  $U_S$  is the expected utility evaluated under the specification  $S$ , and  $Z_t$  is the state variable. For example, if  $S$  is the *SE* setting,  $Z_t$  is the realized level of

<sup>20</sup>[Balduzzi and Lynch \(1999\)](#) and [Campbell and Viceira \(1999\)](#) use a similar definition, but they only calculate U measure, unlike the conditional measure we examine here. [Brennan et al. \(1997\)](#) and [Xia \(2001\)](#) adopt a different measure that calculates the equivalent terminal wealth. [Kandel and Stambaugh \(1996\)](#) use a measure of certainty equivalent return.

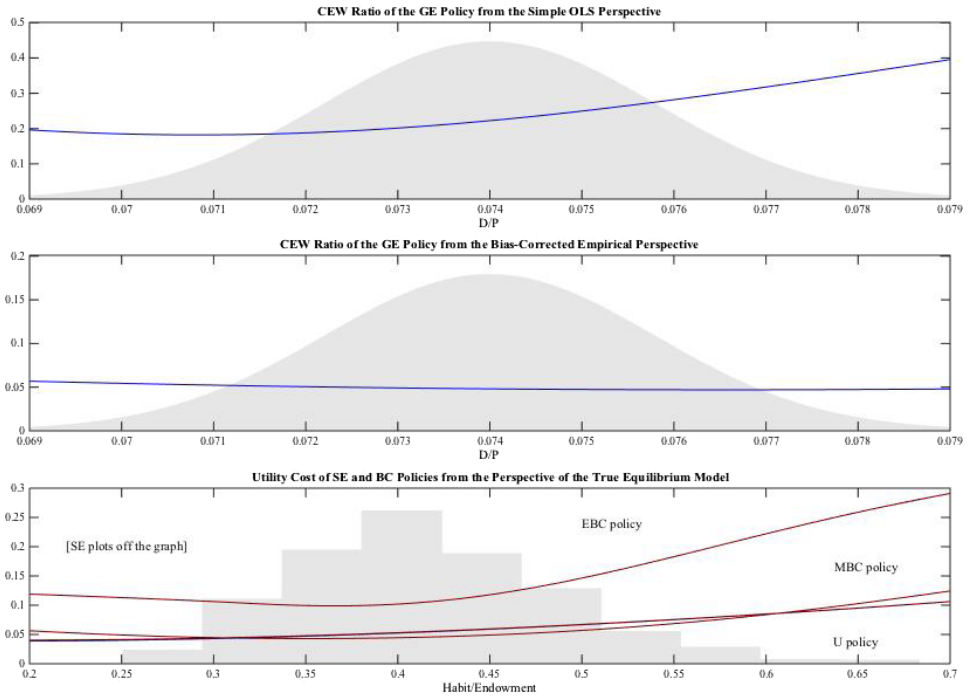


Fig. 13. The utility costs of different consumption/portfolio policies. The top panel shows CEW value (per \$1 of wealth), under SE dynamics, of allowing the investor to switch away from either GE or U policy to SE policy. The middle panel shows CEW value (per \$1 of wealth), under the bias-corrected empirical dynamics, of allowing the investor to switch away from either GE or U policy to the EBC policy. The bottom panel shows the CEW value (per \$1 of wealth), under the true return dynamics, of allowing the investor to switch away from the SE, U, EBC, or MBC policy to the GE policy (the true optimal policy). The rules are computed for a single sample path with the median slope coefficient in the simple regression of excess returns on dividend yields and a planning horizon of 50 years.

dividend yields, and  $\pi_*$  is the U policy, then  $W$  is the maximum fraction of wealth (at  $Z_t$ ) that an investor would be willing to pay to have the opportunity to shift from the U policy to the SE policy (CEW value).

### 5.1. Single path analysis

The top panel of Fig. 13 shows the utility cost, along the path with median predictability under SE specification, of the alternative strategies U and GE.<sup>21</sup> The costs for both policies start close to 20%, at low dividend yield

<sup>21</sup>The utility costs of EBC and MBC are not feasible to calculate in the SE setting because these strategies are time varying and they have to be tracked along with the changes in state variables when calculating expected utility. This is not the case in the equilibrium setting because these strategies may be taken as time invariant in the steady state.

levels, and increase to near 40% of the GE policy and near 50% for the U policy, at high dividend yield levels. Again, following Eq. (14), this means that for low dividend yield levels, \$1 invested in the GE (U) policy provides the same level of utility as \$0.80 invested in SE. This range is consistent with its U expectation reported in the literature (Balduzzi and Lynch, 1999; Campbell and Viceira, 1999). On average, an investor who believes that a simple OLS point estimate is an adequate description of excess return predictability would be indifferent between the true optimal policy and current wealth and the SE policy and a 30% reduction in wealth.

The middle panel of Fig. 13 repeats the utility cost calculation assuming that the empiricist corrects the OLS estimates for the finite-sample bias in the excess return versus dividend yield regressions described in Stambaugh (1999). In two regards, the results are similar to the top panel in Fig. 13. The EBC rule is still more attractive than either the U or GE rule, and the GE and U rules are about equally unattractive to a bias-corrected empiricist. The primary difference between the top and the middle panels of the figure is that, consistent with the results in Fig. 10, the overall utility cost of switching is much lower in this case. In particular, the utility value of \$1 invested in EBC (when that rule is thought to be optimal) is equivalent to \$0.95 invested in either GE or U. This shows that — for this one sample path — EBC and GE (and U) are “close” in a utility cost sense. It remains to be seen whether or not this result holds up across multiple paths.

The utility costs of SE, EBC, and U policies — *viewed from the perspective of the true DGP* — are shown in the bottom panel of Fig. 13.<sup>22</sup> First, for this one path, the utility costs of the SE policy plot off the chart, essentially incurring a utility cost of 100%. The level of exposure to habit risk implied by the SE policy in Fig. 10 is so extreme as to imply that \$1 invested in SE is virtually worthless to the true GE investor. This policy implies such extreme consumption and consumption volatility, that expected wealth tends to zero, when viewed from the perspective of the true conditional moments.

The EBC policy can be implemented in practice, and it yields utility costs ranging from \$0.11 on the dollar (of wealth) to \$0.30 on the dollar. This is clearly a substantial improvement on the SE policy. *However, these*

<sup>22</sup>The true optimal policy is time invariant. In order to determine the utility costs of the suboptimal SE, EBC, MBC and U policies, the initial choice functions were taken as approximations to an infinite horizon solution. Given the 50-year planning horizon of the agent performing the partial equilibrium analysis, there should be very little error in this approximation.

calculations show that the EBC policy still results in a substantial utility cost; *i.e.*, there are significant differences between the VAR assumptions and the true DGP. In particular, the EBC policy does not do nearly as well as another feasible strategy: the Merton–Samuelson (U) policy. The utility costs of U policy are on the order of \$0.07 to \$0.10 on the dollar, indicating that \$1 invested in U policy yields the same level of utility as \$0.90 to \$0.93 invested in the true optimal policy.<sup>23</sup> In fact, the lowest panel in Fig. 13 shows that the utility costs of U policy are essentially indistinguishable from the utility costs of the (infeasible) MBC policy.

## 5.2. Multiple path analysis

The cross-sectional distributions of the utility costs of different rules across the same 500 sample paths used to construct Figs. 11 and 12 are presented in Figs. 14 and 15. The left column of histograms in Fig. 14 compares U and GE policies to SE policy, assuming that the dynamics estimated with simple OLS are an accurate description of the conditional mean excess market return. These densities correspond to the top panel of Fig. 13. The left column of Fig. 14 shows that utility cost estimates, from the perspective of simple OLS dynamics, have a very large cross-sectional variance. The utility costs of the U and GE policies are comparable, and they range from \$0.00 to \$0.60 on the dollar, across the 500 paths. This implies that the results in Fig. 13 are consistent with the overall experience obtained from repeated simulations of the model.

The right column of histograms in Fig. 14 compares U and GE to the bias-corrected empirical policies, under the assumption that the bias-corrected OLS is the correct description of predictability. These plots correspond to the middle panel of Fig. 13, and they show that the empirical bias-correction does change the distribution of the utility costs for U and GE. The utility costs of switching from U or GE to EBC are generally comparable and lower than those in the SE setting, and this is consistent with Fig. 13. The right column of Fig. 14 does suggest that, while the utility cost estimates of switching to U or GE on the median predictability path are close to the mode of the cross-sectional distribution, there are still a large number of paths where the utility costs of U and GE are in excess of \$0.20 on the dollar.

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<sup>23</sup>This is reminiscent of sage advice that is beautifully conveyed in Chinese folklore: “Stand still in a constantly changing uncertain world.”



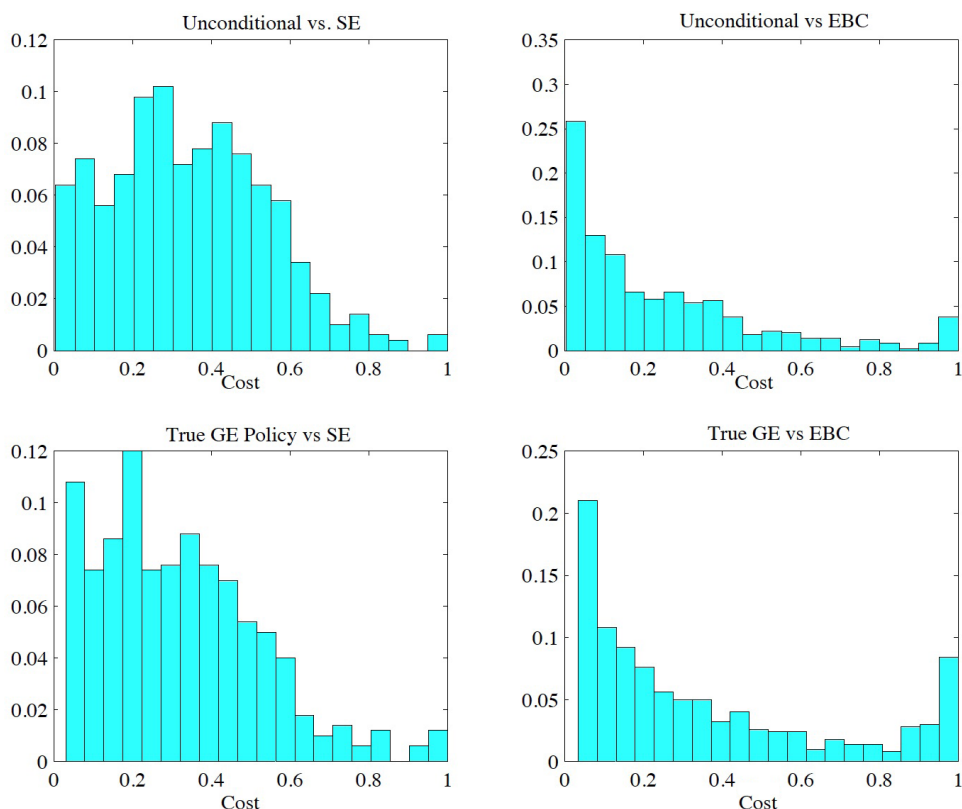


Fig. 14. Utility costs under the empirical dynamics. The left two histograms show the empirical density, measured across 500 independent sample paths, of the utility cost differences between the SE rule and the GE rule and between the SE rule and the U rule, evaluated at the median point in the stationary distribution of the dividend yield. The right two histograms are the analogous plots comparing GE and U rules to the EBC rule, respectively. All of the rules are computed assuming a planning horizon of 50 years.

Figure 15 shows the cross-sectional distribution of the utility costs of SE, EBC, MBC, and U under the true equilibrium dynamics.<sup>24</sup> These histograms are consistent with the single-path results in Fig. 13. In particular, the SE rule is almost always considered an unacceptably risky strategy to the T-S investor. The utility costs are close to \$1 (on the dollar) in more than 80% of the

<sup>24</sup>These distributions contain all the information required to calculate the (frequentist) “risk function” of each policy which is a standard tool for evaluating data-dependent decision rules. See Berger (1985) for a detailed and general treatment of statistical decision theory. Jorion (1986) examines risk functions in a static Markowitz setting, where the investment policies are dependent on the unobservable means of portfolio returns. In a qualitative sense, his findings are very similar to those in this section.

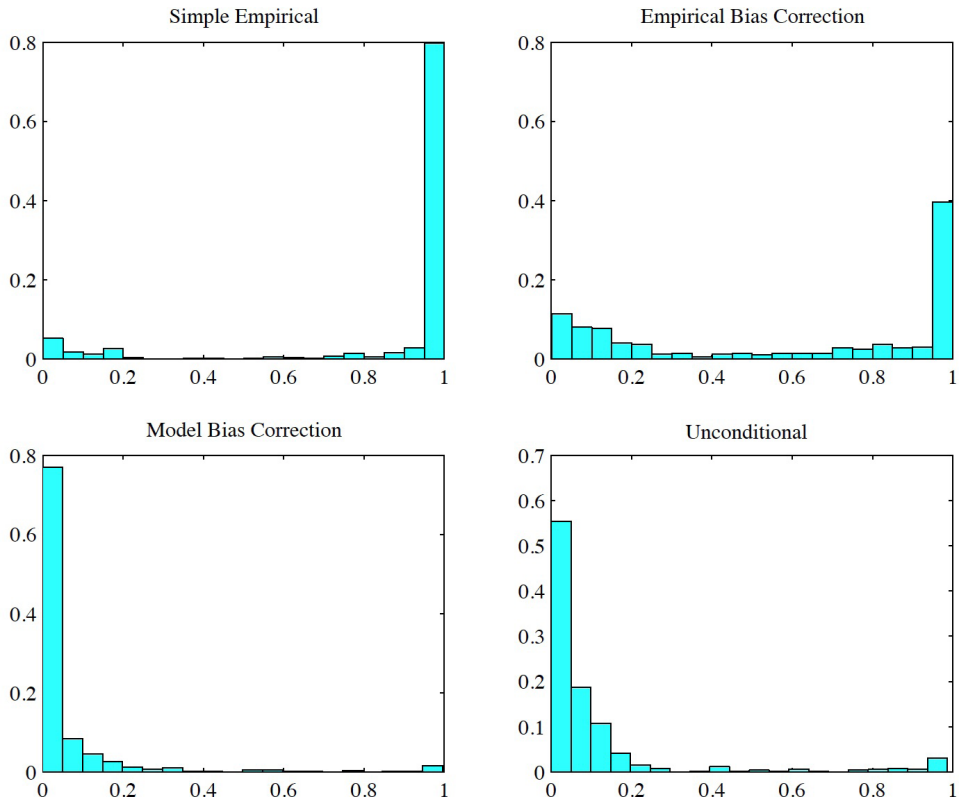


Fig. 15. Utility costs under the true dynamics. The upper-left histogram shows the empirical density, measured across 500 independent sample paths, of the utility cost differences between the true GE rule and the SE rule, evaluated at the median point in the stationary distribution of the dividend yield. Similarly, other histograms show the analogous results for the empirical densities of the utility costs differences between the true GE rule versus the EBC, MBC and U rules, respectively. All rules are calculated assuming a planning horizon of 50 years.

500 paths.<sup>25</sup> The rank ordering of the rules in the bottom panel of Fig. 13 is also generally preserved in the histograms of Fig. 15.

The EBC rule is an improvement on the SE rule, but it still has a modal utility cost (with roughly 40% of the simulations) of \$1.<sup>26</sup> There is a substantial percentage of the simulated paths for which the VAR dynamics

<sup>25</sup>In settings such as ours, it is not uncommon for a large set of policies to result in utility that is essentially  $-\infty$ . Kandel and Stambaugh (1996) restrict attention to asset allocations where wealth invested in the risky asset is between 0% and 99% for precisely this reason (see their Footnote 29). In our setting, investment policies give rise to unbounded negative utility for a variety of reasons. For example, if the consumption rate is too high, wealth will be driven to zero at a rate sufficient to make discounted utility  $-\infty$ .

<sup>26</sup>This is consistent with the number of paths that have negative slopes in the portfolio allocation (see Fig. 12).

underlying EBC are qualitatively different from the true DGP. Again, these differences reflect the variety of differences between the true DGP and the assumptions underlying a VAR with normal errors. The MBC rule is the most attractive to the T-S investor (i.e., has the lowest utility cost in most simulations), with cost levels near 0 in almost 80% of the simulations. U policy also fares very well, although it is slightly more costly than the (infeasible) MBC rule.

Overall, the conclusions that can be drawn from Figs. 14 and 15 are consistent with Fig. 13: Using the SE dynamics gives an extremely distorted and noisy view of the risks and benefits of alternative strategies. Bias correction is useful, but it still fails to improve on U rule or reveal the attractiveness of the true optimal (GE) rule.

## 6. Conclusions

We have examined the properties of estimates of the quantitative significance of asset return predictability using the excess returns and dividend yields generated from an equilibrium model. When the model is calibrated to deliver conditional moments of excess returns and dividend yields that are consistent with the US data, we find that: (i) It is important to bias-correct the simple OLS estimates. This avoids grossly overestimating the effect of predictability on consumption and portfolio rules. (ii) Empirically feasible bias-correction does not solve the problem that conditional moments are difficult to measure. This means that there is a large cross-sectional variance in the estimates of conditional consumption and portfolio rules. (iii) A bias-correction strategy based on the standard VAR assumptions for the DGP for returns — which appears to be a reasonable approximation in the simulated returns — still can generate large utility costs for a significant proportion of the simulated model histories. In fact, under the true dynamics, U policy actually has significantly lower utility costs than either of the empirically feasible conditional policies.

By using a fully specified equilibrium model, we can construct MBC strategies that come very close (under the true dynamics) to recovering GE policies. This final step in our investigation is only possible because our model economy fully specifies the Arrow–Debreu prices, and consequently, the joint dynamics of dividend yields and asset returns. It is a fundamental advantage of a GE model over a reduced form DGP, such as a VAR. While some of the problems with empirically estimated decision rules may be addressed with exogenously specified returns, the severity of these biases — and their

connection to the explicit source of return predictability — cannot be understood in the absence of a fully specified model.

The conclusions from this study apply most directly to calibration studies. Their implications for Bayesian analyses of return predictability are less clear, and a thorough answer to this question is beyond the scope of this paper. An examination of this kind would require a thorough treatment of the choice of prior distribution. The results in this paper — particularly the comparatively little information about the conditional mean contained in realistic samples — suggest that the choice of the prior distribution has non-trivial implications for the properties of the Bayesian estimates of optimal rules. Simple priors, motivated by issues of tractability of the likelihood function, may prove to be inadequate in capturing the true nature of uncertainty about return predictability.

More fundamentally, however, a truly Bayesian perspective would have profound implications for the underlying model structure, since we would want to match the agent’s problem in the partial equilibrium analysis with the problem solved in GE. For example, the infinite-horizon, constant parameter, Markov structure of the model that we use here suggests that model uncertainty should disappear after a finite amount of time. Therefore, a more complicated data-generating structure would need to be constructed to ensure uncertainty in the steady-state of the model. In addition, the underlying concept of the model equilibrium must be extended to state explicitly the nature and evolution of the beliefs of each agent type. This is a challenging problem, and it is expected to introduce additional components driving the dynamics of model-generated returns. This analysis remains as an important direction for future.

### Appendix A. Proof of Proposition

The social planner’s problem, starting at time  $t$ , is

$$\max_{C_{1,s}, C_{2,s}} E_t \left[ \int_t^\infty \exp \left( -\beta(s-t) \left( \lambda \frac{C_{1,s}^{1-\gamma_1} - 1}{1-\gamma_1} + (1-\lambda) \frac{(C_{2,s} - \nu X_s)^{1-\gamma_2}}{1-\gamma_2} \right) \right) ds \right], \tag{A.1}$$

where  $\lambda$  is a constant in  $(0, 1)$ , subject to the resource constraint

$$C_{1,s} + C_{2,s} = Y_s, \tag{A.2}$$

for all  $s > t$ . Since there are no opportunities in this economy for intertemporal resource transfer, the dynamic problem in (A.1) is equivalent

to solving

$$\max_{C_{1,t}, C_{2,t}} \left\{ \lambda \frac{C_{1,t}^{1-\gamma_1} - 1}{1 - \gamma_1} + (1 - \lambda) \frac{(C_{2,t} - \nu X_t)^{1-\gamma_2} - 1}{1 - \gamma_2} \right\}, \quad (\text{A.3})$$

at all dates  $t$ , subject to the constraint (A.2).

Let the Lagrange multiplier for the resource constraint (A.2) be denoted  $\alpha_t$ , then the first order conditions to the problem in (A.3) are

$$\lambda C_{1,t}^{-\gamma_1} = \alpha_t \quad (\text{A.4})$$

and

$$(1 - \lambda)(C_{2,t} - \nu X_t)^{-\gamma_2} = \alpha_t, \quad (\text{A.5})$$

along with (A.2). So,

$$C_{1,t} = \left( \frac{\alpha_t}{\lambda} \right)^{-1/\gamma_1} \quad (\text{A.6})$$

and

$$C_{2,t} = \left( \frac{\alpha_t}{1 - \lambda} \right)^{-1/\gamma_2} + \nu X_t, \quad (\text{A.7})$$

and, together with (A.2), we have

$$\left( \frac{\alpha_t}{\lambda} \right)^{-1/\gamma_1} + \left( \frac{\alpha_t}{1 - \lambda} \right)^{-1/\gamma_2} + \nu X_t = Y_t.$$

If  $\gamma_1 = \gamma_2 = \gamma$ , then

$$\alpha_t = \lambda \left[ 1 + \left( \frac{\lambda}{1 - \lambda} \right)^{-\frac{1}{\gamma}} \right]^\gamma (Y_t - \nu X_t)^{-\gamma}. \quad (\text{A.8})$$

Plugging (A.8) into (A.6) and (A.7) implies that the consumption shares of the two agent types can be written as

$$\frac{C_{1,t}}{Y_t} = \left[ 1 + \left( \frac{\lambda}{1 - \lambda} \right)^{-\frac{1}{\gamma}} \right]^{-1} \omega_t \quad (\text{A.9})$$

and

$$\frac{C_{2,t}}{Y_t} = \left( \frac{\lambda}{1 - \lambda} \right)^{-\frac{1}{\gamma}} \left[ 1 + \left( \frac{\lambda}{1 - \lambda} \right)^{-\frac{1}{\gamma}} \right]^{-1} \omega_t + [1 - \omega(t)], \quad (\text{A.10})$$

where

$$\omega_t \equiv 1 - \nu \frac{X_t}{Y_t},$$

for notational convenience. Taken together, (A.9) and (A.10) demonstrate the statement in the proposition regarding consumption shares.

Now, the price of the risky asset — which is a claim to the aggregate endowment — can be calculated using the marginal rate of inter-temporal substitution of either agent, at the equilibrium consumption levels. For convenience, we will use the T-S Agent’s utility function:

$$S_t^1 = E_t \left[ \int_t^\infty \exp(-\beta(s-t)) \left( \frac{C_{1,s}}{C_{1,t}} \right)^{-\gamma} Y_s ds \right], \quad (\text{A.11})$$

which can be rewritten, using (A.9), as

$$S_t^1 = [\omega_t Y_t]^\gamma E_t \left[ \int_t^\infty \exp(-\beta(s-t)) \omega_s^{-\gamma} Y_s^{1-\gamma} ds \right].$$

By Assumption A4,  $Y_t$  provides no information about the future evolution of  $Y_s$  (or  $f(Y_s)$ ), and  $S_t^1$  is a function of only  $\omega_s$  for  $s > t$ . The only effect of the current endowment,  $Y_t$  is to scale the current price,  $S_t^1$ , up or down. This implies that the expected return to the risky asset over the interval from  $t$  to  $t + \tau$ , defined as

$$\frac{1}{S_t^1} E_t \left[ S_{t+\tau}^1 + \int_t^{t+\tau} Y_u du \right],$$

is independent of  $Y_t$ . As noted earlier, closed-form solutions for the asset prices and expected returns do not exist.

The price of a risk-free asset maturing at  $s > t$  can be constructed in a manner that is analogous to Eq. (A.11):

$$S_{s-t,t}^0 = E_t \left[ \exp(-\beta(s-t)) \left( \frac{C_{1,s}}{C_{1,t}} \right)^{-\gamma} ds \right],$$

which reduces to

$$S_{s-t,t}^0 = [\omega_t Y_t]^\gamma E_t [\exp(-\beta(s-t)) \omega_s^{-\gamma} Y_s^{-\gamma} ds],$$

and the dependence of the risk-free term structure on the endowment is, again, only a scaling effect.

The results in this proposition, for this economy, are fundamentally the same as those derived by Chan and Kogan (2002) in a “keeping up with the Joneses” habit model with a geometric Brownian motion endowment. See, in particular, their Lemmas 1 and 2.

## Appendix B. Solving the Model and Simulating Asset Returns

A solution to the model economy described in Sec. 2 consists of asset prices,  $\{S^0, S^1\}$  and portfolio rules  $\{\theta_{i,0}, \theta_{i,1}\}$  for  $i = 1, 2$ , that are time-invariant functions of the state variables (the aggregate endowment (dividend) and the level of the habit index) and that implement an equilibrium in the model. Since explicit, closed-form solutions for these functions do not exist, we approximate them numerically.

At the parameter values that we have chosen, the second welfare theorem applies, and we make use of a Social Welfare Function to characterize the equilibrium consumption allocation.<sup>27</sup> Given a utility weight,  $\lambda$ , the *social planner's problem* is defined as:

$$V\left(Y_t, \frac{X_t}{Y_t}\right) = \max_{\{C_{1,t}, C_{2,t}\}_{t=1}^{\infty}} \left\{ \lambda U_1(C_{1,t}) + (1 - \lambda) U_2(C_{2,t}, X_t) + \beta E \left[ V\left(Y_{t+1}, \frac{X_{t+1}}{Y_{t+1}}\right) \right] \right\}, \quad (\text{B.1})$$

subject to the aggregate budget constraint (goods market clearing),

$$C_{1,t} + C_{2,t} = Y_t,$$

for all  $t$ .

The dynamic programming problem (B.1) is solved using the methods described in Kushner and Dupuis (1992, Sec. 5.3).<sup>28</sup> Their approach is to discretize the underlying state space using a lattice and then form a Markov chain on the discretized states that approximates the continuous dynamics. In our setting, the state space is the set of endowment and habit/endowment pairs,  $(Y, X/Y) \in \mathbb{R}^2$ , and the discretized state space is a  $150 \times 150$  lattice formed on the logarithmic transformation of these pairs.<sup>29</sup> Transition probabilities at points within the discretized state space are to neighbors only and this feature of the numerical approximation is the key to reducing the computational burden.

Once the approximating Markov chain is formed, standard numerical dynamic programming procedures can be applied. One particularly efficient

<sup>27</sup> Recall the discussion in Sec. 2.2.

<sup>28</sup> See Hindy *et al.* (1997a,b) for another application of these solution methods.

<sup>29</sup> The state variable for the endowment process,  $Y$ , is unbounded and in the discretized state space this state variable is truncated at a high, but finite, level. Kushner and Dupuis (1992) give conditions under which the solution of a dynamic programming problem on the bounded approximating Markov chain converges to the solution on the unbounded continuous state space. We use these convergence results to guide our choice of the upper bound for  $Y$  in our numerical solution.

algorithm is referred to as *policy function iteration*, see [Puterman \(1994\)](#). It is instructive to consider policy iteration within the context of the social planner’s problem. The algorithm starts with an arbitrary allocation, represented by a consumption share in each state. Given this allocation, discounted social utility for the policy can be calculated, as can the marginal social utility. A “policy improvement” step can be made by solving the first-order conditions for the social planner’s problem, taking the continuation value as given, and the newly derived allocation can be used for the next step in the algorithm. This recursive solution method continues until the convergence criteria are satisfied.

The policy iteration algorithm yields the solution to (B.1) and is a pair of time-invariant policy functions that define the optimal consumption of the two agent types as functions of the exogenous state variable ( $Y$ ) and the endogenous state variable ( $X/Y$ ). Given these consumption processes, asset prices are constructed using the state-prices defined from the inter-temporal marginal rate of substitution evaluated at the optimal consumption values. Since this is a complete-markets equilibrium, the inter-temporal marginal rates of substitution for the two agent types will be equalized at all dates and states. Therefore, we solve for asset prices using the prices defined in terms of the utility function of the T-S Agent.

Specifically, the price of the risk-free asset and the risky asset are calculated as:

$$S^0\left(Y_t, \frac{X_t}{Y_t}\right) = E\left[\beta \frac{U'_1(C_{1,t+1}^*)}{U'_1(C_{1,t}^*)}\right], \quad (\text{B.2})$$

and

$$S^1\left(Y_t, \frac{X_t}{Y_t}\right) = E\left[\beta \frac{U'_1(C_{1,t+1}^*)}{U'_1(C_{1,t}^*)} \left(Y_{t+1} + S_1\left(Y_{t+1}, \frac{X_{t+1}}{Y_{t+1}}\right)\right)\right]. \quad (\text{B.3})$$

Asset prices and returns are calculated by solving for the fixed point implicit in this pricing equation. The expectations operator is calculated using the transition matrix for the Markov chain describing the endowment process and defined on the discretized state space.

The artificial data used in [Sec. 3](#) were generated by first simulating a sequence of draws of the endowment growth rate using the function `normrnd` in Matlab. Given, initial values for the endowment and the habit index, interpolations of the optimal policy functions (using the command `interp2` in Matlab) were used to generate the consumption of the agent types and



(implicitly) the habit index, and the asset data was constructed using Eqs. (B.2) and (B.3).

### Appendix C. Computing the Optimal Policy in Partial Equilibrium

In the empirical literature on return predictability and the optimal consumption/portfolio choice problem, a standard statement of the consumer's problem is:

$$\max_{\{C_t, \alpha_t\}_{t=0}^T} E_0 \left[ \sum_{t=0}^T \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} \right], \quad (\text{C.1})$$

subject to the budget constraint

$$W_{t+1} = (W_t - C_t)[\alpha_t(R_{t+1}^1 - R_t^0) + R_t^0], \quad (\text{C.2})$$

where  $\beta$  and  $\gamma$  have the same interpretation as in Sec. 2 and  $W_t$  is the (stochastic) wealth of the agent held on arrival at date  $t$ . Consistent with the model in Sec. 2,  $R_{t+1}^1$  is the return to the risky asset from  $t$  to  $t+1$  and  $R_t^0$  is the risk-free return from  $t$  to  $t+1$ , and it is known at time  $t$ . The consumer chooses the consumption level,  $C_t$ , and the share of wealth held in the risky asset,  $\alpha_t$ .<sup>30</sup>

If the agent's portfolio return between  $t$  and  $t+1$  is defined as

$$R_{t+1}^p \equiv \alpha_t(R_{t+1}^1 - R_t^0) + R_t^0$$

and  $c_t \equiv C_t/W_t$  is the consumption share of wealth, then Eq. (C.2) can be rewritten as

$$W_{t+1} = (1 - c_t)W_t R_{t+1}^p. \quad (\text{C.3})$$

In general, the solution to the consumer's problem will consist of consumption and portfolio rules that are functions of the investor's wealth, which serves as the sole state variable, when the problem is stated in its recursive form as

$$V_t(W_t) = \max_{c, \alpha} \left\{ \frac{c_t^{1-\gamma} W_t^{1-\gamma}}{1-\gamma} + \beta E[V_{t+1}(W_{t+1}) | W_t] \right\}. \quad (\text{C.4})$$

<sup>30</sup>There is actually some variation in the form of the portfolio problem solved in the different papers cited in the literature. Some authors do not solve the problem allowing for consumption at intermediate dates, and Brennan *et al.* (1997) solve a continuous-time version of the problem. The specification in Eqs. (C.1) and (C.2) corresponds most directly to Balduzzi and Lynch (1999) and Brandt (1999).

$V_t(W_t)$  is the indirect utility function, and (C.4) is the Bellman equation of the dynamic programming version of the problem (and  $V_t(W_t)$  is also called the value function). When the investor's horizon is infinite and the model is time-homogeneous, as we will assume, the value function no longer has an explicit dependence on  $t$ .

Samuelson (1969) discusses the solution for a problem of the general form of (C.1) and (C.2).<sup>31</sup> In the absence of return predictability, the portfolio allocation is a constant fraction of wealth. This fraction depends on both the agent's risk tolerance and the market-determined (constant) trade-off between the excess return and the risk of the portfolio. If returns follow a Markov process whose transition density depends on a state variable (or vector of state variables)  $Z_t$ , then investor's Bellman equation becomes

$$V_t(W_t, Z_t) = \max_{c, \alpha} \left\{ \frac{c_t^{1-\gamma} W_t^{1-\gamma}}{1-\gamma} + \beta E[V_{t+1}(W_{t+1}, Z_{t+1}) | W_t, Z_t] \right\}, \quad (\text{C.5})$$

and the solution is a pair of (time-invariant) policy functions,  $c(W_t, Z_t)$  and  $\alpha(W_t, Z_t)$  that depend on both wealth and the state variable that predicts future returns.

In particular,

$$V_t(W_t, Z_t) = \frac{(c_t^*)^{-\gamma} W_t^{1-\gamma}}{1-\gamma},$$

where  $c_t^*$  is the optimal consumption ratio for the investor at time  $t$  and satisfies

$$c_t^* = [1 + (\beta E[(R_{t+1}^p)^{1-\gamma} (c_{t+1}^*)^{-\gamma} | Z_t])^{1/\gamma}]^{-1}. \quad (\text{C.6})$$

The optimal portfolio choice  $\alpha_t^*$  solves the following equation

$$E[(c_{t+1}^* R_{t+1}^p)^{-\gamma} (R_{t+1}^1 - R_t^0) | Z_t] = 0. \quad (\text{C.7})$$

(Recall that  $R_{t+1}^p = \alpha_t(R_{t+1}^1 - R_t^0) + R_t^0$ .) In general, the solutions to these equations are not available, and  $c_t^*$  and  $\alpha_t^*$  must be approximated numerically.

In order to calculate the optimal consumption and portfolio choices, we need to solve Eqs. (C.6) and (C.7) recursively. The boundary condition is  $c_T^* = 1$  and  $\alpha_T^* = 0$ . At each time  $t$ , we solve (C.6) first to obtain the solution for  $\alpha_t^*$ , and then plug it into (C.7) to solve for  $c_t^*$ . The expectation is taken

<sup>31</sup>Merton (1969) is the companion paper that considers the continuous-time limit of the discrete-time problem.

under a conditional probability distribution function  $p(R_{t+1}, Z_{t+1} | Z_t)$ , which is assumed to be known (and normal in most of the predictability studies).

Brandt (1999) uses similar first-order conditions in a non-parametric approach to solving the portfolio choice problem fit to empirical data. Barberis (2000) uses a brute-force approach in solving the dynamic programming problem through massive simulations. Barberis contains a detailed description of the procedure he uses to investigate the portfolio choice problem for an investor maximizing only the utility of terminal wealth. Balduzzi and Lynch (1999) use a backward induction procedure to solve the portfolio choice problem, although they do not provide details of the procedure. Both papers rule out borrowing and short-sales by restricting portfolio choice to lie inside  $(0,1)$ . This is primarily for computational convenience, and yet this restriction is not a constraint faced by the investor in solving the optimal consumption/investment problem. It is demonstrated, both here and in Brandt (1999), that this restriction is frequently violated when it is not imposed. In solving the first-order conditions (C.6) and (C.7), we will not impose these constraints.

The non-linear structure of (C.7) poses a challenge for solving  $\alpha_t^*(Z_t, T)$  directly. In solving for the optimal portfolio choice, the following iteration algorithm is adopted, based on the observation that the portfolio choice for the same state does not change too much from period to period. If the optimal choice at  $t+1$  is  $\alpha_{t+1}^*(Z_t, T)$ , then the choice at  $t$  is  $\alpha_t(Z, T) = \alpha_{t+1}^*(Z, T) + \varepsilon(Z)$  that supposedly satisfies (C.7).

Using a first-order Taylor-series expansion to linearize in  $\varepsilon(Z)$ , one can immediately get an approximation for  $\varepsilon(Z)$ , which can be obtained from

$$E[(c_{t+1}^*(\alpha_{t+1}^*(\tilde{R}_{t+1} - R_{ft}) + R_{ft}))^{-\gamma}(\tilde{R}_{t+1} - R_{ft}) | Z_t] \\ \simeq \gamma \varepsilon_t(Z_t) E \left[ (c_{t+1}^*(\alpha_{t+1}^*(\tilde{R}_{t+1} - R_{ft}) + R_{ft}))^{-\gamma} \frac{(\tilde{R}_{t+1} - R_{ft})^2}{\alpha_{t+1}^*(\tilde{R}_{t+1} - R_{ft}) + R_{ft}} \middle| Z_t \right].$$

Then replacing  $\alpha_{t+1}^*(Z, T)$  with  $\alpha_t(Z, T)$ , one can continue this iteration and arrive at the solution. For a reasonable range of parameters, this algorithm converges very quickly.

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