Portfolio Choice with Market Closure and Implications for Liquidity Premia*

Min Dai, Peifan Li, Hong Liu, and Yajun Wang

This version: October, 2014

Abstract

Most existing portfolio choice models ignore the prevalent periodic market closure and the fact that market volatility is significantly higher during trading periods. We find that market closure and the volatility difference across trading and nontrading periods significantly change optimal trading strategies. In addition, we numerically demonstrate that transaction costs can have a first order effect on liquidity premia that is largely comparable to empirical findings. Moreover, this effect on liquidity premia increases in the volatility difference, which is supported by our empirical analysis.

Journal of Economic Literature Classification Numbers: G11, D11, D91, C61.

Keywords: Portfolio Choice, Market Closure, Volatility Dynamics, Liquidity Premia.

---

*Dai is from Department of Mathematics and Risk Management Institute of National University of Singapore (NUS), Li is from Department of Mathematics of NUS, Liu is from Olin Business School of Washington University in St. Louis and CAFR, and Wang is from Robert H. Smith School of Business of University of Maryland. We thank Jerome Detemple (the editor), two anonymous referees, John Campbell, George Constantinides, Phil Dybvig, Bob Geske, Mark Loewenstein, Richard Roll, Eduardo Shwartz, Pietro Veronesi, Guofu Zhou and seminar participants at Oxford University, Stockholm School of Economics, UCLA, UIUC, University of Texas at Dallas, and Washington University in St. Louis for helpful comments. We are grateful to Chen Yang for his excellent research assistance. Min Dai acknowledges support from Singapore AcRF grant R-146-000-188/201-112 and NUS Global Asia Institute - LCF Fund R-146-000-160-646. Authors can be reached at matdm@nus.edu.sg, lipeifan@nus.edu.sg, liuh@wustl.edu, and ywang22@rhsmith.umd.edu.

Market closures during nights, weekends, and holidays are implemented in almost all financial markets. An extensive literature on stock return dynamics across trading and nontrading periods finds that, while expected returns do not vary significantly across these periods, return volatility is much higher during trading periods (e.g., French and Roll (1986), Stoll and Whaley (1990), Tsiakas (2008), see Figure 2). For example, French and Roll (1986) and Stoll and Whaley (1990) find that return volatility during trading periods is more than four times the volatility during non-trading periods on a per-hour basis.[1] However, most of the existing portfolio selection models assume that market is continuously open and return volatilities are the same across trading and nontrading periods.[2] Therefore, the practical relevance of the optimal trading strategy obtained in these models can be limited. In addition, one of the important implications of this assumption is that the effect of transaction costs on liquidity premium is too small to match empirical evidence (e.g., Constantinides (1986), Jang, Koo, Liu and Loewenstein (2007)).

In this paper, we consider a continuous-time optimal portfolio choice problem of a small investor who can trade a riskfree asset and a risky stock that is subject to proportional transaction costs. Different from the standard literature, we assume market closes periodically and stock return dynamics may differ across trading and nontrading periods. We show the existence, uniqueness, and smoothness of the optimal trading strategy. We derive a closed-form solution in the absence of transaction costs. In the presence of transaction costs, we explicitly characterize the solution to the investor's problem and derive certain helpful comparative statics on the optimal trading strategies. We find that in the absence of transaction costs, the investor almost always trades at market close and market open if Sharpe ratios vary across trading and nontrading periods. In the presence of even small transaction costs, however, he trades only infrequently.

[1]French and Roll (1986) conclude that the principle factor behind high trading-time variances is the private information revealed by informed trades during trading hours, although mispricing also contributes to it.

[2]See, for example, Merton (1971), Constantinides (1986), Vayanos (1998)

This paper also contributes to the literature on the effect of transaction costs on liquidity premia. As explained by Constantinides (1986), consider two assets with perfectly correlated rates of return and equal variance, where the first asset is subject to proportional transaction cost but the second is not. Then if both assets are held in equilibrium, the expected return on the first asset must exceed that of the liquid counterpart by some liquidity premium. Following Constantinides (1986), we define liquidity premium as the maximum expected return an investor is willing to exchange for zero transaction cost. Liquidity premia found by most theoretical portfolio selection models using this measure are well below empirical findings. For example, Constantinides (1986) finds that the liquidity premium to (round-trip) transaction cost (LPTC) ratio is only about 0.07 for a proportional (round-trip) transaction cost of 1%, while Amihud and Mendelson (1986) find that the LPTC ratio is about 1.9 for NYSE stocks in their empirical study. Using a regime switching model, Jang, Koo, Liu and Loewenstein (2007) show that the LPTC ratio given reasonable calibration is about 0.25. Lynch and Tan (2011) show that incorporating return predictability, state-dependent transaction costs and wealth shock can generate greater liquidity premia than Constantinides (1986). However, the liquidity premia found by Lynch and Tan (2011) with reasonable parameter values are still significantly smaller than the corresponding empirical evidence. We numerically demonstrate that if one incorporates the well-established fact that market volatility is significantly higher during trading periods, then transaction costs can have a first order effect that is comparable to empirical evidence. For example, when the volatility during trading periods is three (resp., four) times that during nontrading periods, the LPTC ratio in our model is about 1.76 (resp., 2.22) for a proportional round-trip transaction cost of 1%, more than 20 times higher than what Constantinides (1986) finds. This result is not sensitive to a three-period extension with an after-hour trading period in addition to day time trading period and overnight market closure period. The main intuition for why the liquidity premium is much higher in our model is simple: The opportunity cost of not being able to rebalance costlessly to take advantage of the time-varying return dynamics is much greater when return dynamics changes significantly and frequently.

Transaction costs decrease an investor's utility through two channels: First, the wealth is reduced by transaction cost payment; second, the investor cannot always trade to maintain the optimal risk exposure. Liquidity premia found in the existing literature (e.g., Constantinides (1986), Jang et. al. (2007)) mainly come from the transaction cost payment channel. Surprisingly, we find that the significantly higher liquidity premium in our model mainly comes from the substantially "suboptimal" risk exposure chosen to control transaction costs.

While it is beyond the scope of this paper to provide an equilibrium model that can generate different Sharpe ratios across day and night as what is observed in data, such an equilibrium can be consistent with an economy with heterogeneous investors, e.g., some investors may be more risk averse toward carrying overnight inventories than others, they may have different time discount rates, or they may have heterogeneous beliefs on the time varying return dynamics.[3] As we have mentioned above, because of the presence of transaction costs, investors trade infrequently even when the overnight Sharpe ratio is much greater. This suggests that small heterogeneity may be sufficient to sustain an equilibrium with different Sharpe ratios across trading and nontrading periods. In this paper, we take the salient and robust volatility and Sharpe ratio patterns across trading and nontrading periods that are found by a large literature as a given equilibrium outcome and consider what is the impact of illiquidity on a *small* investor who does not have any price impact.

[3] We solved a 4-period discrete time equilibrium model with market closure and transaction costs, the results of which are not reported in the paper to save space. This simple model illustrates that with heterogeneous agents in an economy, without transaction cost, market closure can cause large trades at market closure and opening time, as in the no transaction cost case of our paper. In addition, the presence of transaction costs can significantly reduce trading sizes, as in the case with transaction costs in our paper.

Our model suggests that conditional on the same increase in the transaction costs (e.g., from 0 to 1%), stocks with greater volatility variation across trading periods and nontrading periods require higher additional liquidity premia. Indeed, our empirical analysis using the methodology of Eleswarapu (1997) finds that liquidity premia are higher for stocks with greater volatility-differences across trading and nontrading periods. More specifically, we examine the cross-sectional relation between excess return and spread using the Fama-MacBeth type regressions on equally-weighted portfolios from triple-sorting by average volatility difference $\sigma_d - \sigma_n$ across trading and nontrading periods in the previous year, average relative bid-ask spreads in the previous year, and their estimated betas in the last three years. We find that spreads significantly affect excess returns. Indeed, consistent with the findings of the existing literature (e.g., Amihud and Mendelson (1986), Eleswarapu (1997)), the highly significant coefficient of *Spread* implies a 1% increase in the spread is associated with a 0.22% increase in the monthly risk-adjusted excess return. However, we find that this significant impact of transaction costs mainly comes from stocks with high volatility differences across trading and nontrading periods. For example, for a 1% increase in the spread, stocks with high volatility-differences require 0.36% higher monthly risk adjusted excess return than those with low volatility-differences. We further demonstrate that volatility variation across trading and nontrading periods is still an important determinant of liquidity premium after controlling for firm size, book-to-market ratio, trading volume, and portfolio loadings on Fama-French factors and Carhart four-factors. As far as we know, this is the first empirical analysis that indicates that volatility difference across trading and non-trading periods significantly affects liquidity premia.

With regard to the findings on liquidity premia, the closest work to this model is Jang et. al. (2007). There are several important differences, however. First, this model can generate liquidity premium that is comparable to empirical findings. Since Jang et. al. (2007) rely on regime switching between bear and bull markets and the historical switching frequency is low, for a reasonable calibration of their model they can only generate a LPTC ratio of 0.25, which is significantly lower than those found by empirical studies. Second, we also conduct empirical analysis to test whether as suggested by our model, volatility variation across trading and nontrading periods is an important determinant of liquidity premium. Third, this is the first paper to numerically demonstrate that significantly suboptimal trading strategy caused by transaction cost can be the main driving force behind a greater liquidity premium. For example, in Jang et. al. (2007), greater liquidity premium mainly comes from the higher transaction cost payment caused by greater trading frequency. Finally, since in Jang et. al. (2007), regimes switch at random times, one needs a good estimation of the switching frequency and estimation error may affect the accuracy of the liquidity premium estimation. In contrast, in this model, market closes and opens at mostly known deterministic times and therefore there is no estimation error of the switching times.

This paper is also related to equilibrium models with market closure or with transaction costs. Hong and Wang (2000) consider an equilibrium model with periodic market closure and CARA investors.[4] They find that the equilibrium volatility during trading periods can be higher. They also show that closures can make prices more informative about future payoffs. Different from our model, they do not consider the presence of transaction costs. In an equilibrium model with one shot market closure, Longstaff (2009) examines the effect of one nontrading period on asset prices. Consistent with empirical evidence and our model, he finds that the value of liquidity can represent a large portion of the equilibrium price of an asset. Assuming market is continuously open, Vayanos

[4]For CARA utility, it is still feasible to solve the investor's problem for our model. The main difference from the optimal trading strategy for the CRRA case without transaction cost is that it is optimal to invest a constant dollar amount (instead of a constant fraction of wealth) in stock. With transaction costs, there is a time-varying no-transaction interval for the dollar amount invested in stock, out of which it is optimal to buy or sell to the closest boundary. As most of the existing literature, we do not consider the CARA case as the main case because there is no wealth effect on the dollar amount invested in the stock.

(1998) finds transaction costs have small impact on asset returns. Assuming quadratic transaction costs, which implies small cost for small trades, Heaton and Lucas (1996) find significant liquidity premium only in the presence of large transaction costs. Both of these models assume i.i.d. returns over time.

The rest of the paper is organized as follows. Section 1 presents the model with transaction cost, market closure, and different return dynamics across trading and nontrading periods. Section 2 solves the case without transaction costs as a benchmark for later comparison. Section 3 provides characterizations of the solution and some comparative statics for the optimal trading strategy. Numerical and graphical analysis is presented in Section 4. In Section 5, we extend to a three-period model: a period with no trade at all, a period with trade but high transaction costs, and a period with regular trading. In Section 6, we empirically examine whether the volatility difference across trading and no trading periods is important in affecting liquidity premia. Section 7 closes the paper. All proofs are presented in the Appendix.

1. The model

We consider an investor who maximizes his constant relative risk averse (CRRA) utility from terminal liquidation wealth at $T \in (0, \infty)$. The investor can invest in two financial assets. The first asset ("bond") is riskless, growing at a continuously compounded, constant rate r. The second one is risky ("stock"). Different from the standard literature, we assume that the stock market closes and opens periodically. Specifically, the investment horizon T is partitioned into $0 = t_0 < ... < ... < t_{2N+1} = T$. Market is open in time intervals $[t_{2i}, t_{2i+1}]$ ("day"); while the market is closed and thus no trading takes place in $(t_{2i+1}, t_{2i+2}), \forall i = 0, 1, ..., N$ ("night").[5] When market is open, the investor can buy the stock at the ask price $S_t^A = (1 + \theta)S_t$ and sell the stock

[5]These intervals can be of different length, and thus can deal with closure on weekends and holidays.

at the bid price $S_t^B = (1-\alpha)S_t$, where $\theta \geq 0$ and $0 \leq \alpha < 1$ represent the proportional transaction cost rates and S_t evolves continuously across day and night as

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma(t)d\mathcal{B}_t, \tag{1}$$

with

$$\mu(t) = \begin{cases} \mu_d, & \text{day} \\ \mu_n, & \text{night} \end{cases} \quad \text{and} \quad \sigma(t) = \begin{cases} \sigma_d, & \text{day} \\ \sigma_n, & \text{night,} \end{cases}$$

where $\mu_d > r, \mu_n > r, \sigma_d > 0, \sigma_n > 0$ are assumed to be constants and $\{\mathcal{B}_t; t \geq 0\}$ is a one-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with $\mathcal{B}_0 = 0$ almost surely. We assume $\mathcal{F} = \mathcal{F}_\infty$, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous and each $\mathcal{F}_t$ contains all null sets of $\mathcal{F}_\infty$.

When $\alpha + \theta > 0$, the above model gives rise to equations governing the evolution of the dollar amount invested in the bond, x_t, and the dollar amount invested in the stock, y_t:

$$dx_t = rx_t dt - (1+\theta)dI_t + (1-\alpha)dD_t, \tag{2}$$

$$dy_t = \mu(t)y_t dt + \sigma(t)y_t d\mathcal{B}_t + dI_t - dD_t, \tag{3}$$

where the cumulative stock sales process D and purchases process I are adapted, nondecreasing, and right continuous with $D(0) = I(0) = 0$ and both dI_t and dD_t are restricted to be 0 during night.

Let x_0 and y_0 be the given initial positions in the bond and the stock respectively. We let $\mathcal{A}(x_0, y_0)$ denote the set of admissible trading strategies (D, I) such that (2) and (3) hold, and the investor is always solvent, i.e.,

$$W_t \geq 0, \quad \forall t \geq 0, \tag{4}$$

where

$$W_t = x_t + (1-\alpha)y_t^+ - (1+\theta)y_t^- \tag{5}$$

is the time t wealth after closing the stock position. Because the investor cannot trade when market is closed and the stock price can get arbitrarily close to 0 and is unbounded above, solvency constraint (4) implies that the investor cannot borrow or shortsell at market close.

The investor's problem is then

$$\sup_{(D,I)\in\mathcal{A}(x_0,y_0)} E\left[u(W_T)\right], \tag{6}$$

where the utility function is given by

$$u(W) = \frac{W^{1-\gamma}}{1-\gamma}$$

and $\gamma > 0$ is the constant relative risk aversion coefficient.[6]

2. Optimal trading without transaction costs

For purpose of comparison, we first consider the case without transaction costs (i.e., $\alpha = \theta = 0$). In this case when the market is open, the standard Hamilton-Jacobi-Bellman (HJB) equation holds and it is optimal to continuously trade. The basic idea for solving the investor's problem is to solve it backward iteratively for the last day, then the last night, and then the next-to-last day, so on and so forth.

Let $\pi_t = \frac{y_t}{x_t+y_t}$ be the fraction of wealth invested in the stock at time t and π_M ("Merton line") be the optimal fraction of wealth invested in the stock in the absence of market closure and transaction costs. Then it can be shown that

$$\pi_M(t) = \frac{\mu(t)-r}{\gamma\sigma(t)^2}, \quad \forall t \in [0,T]. \tag{7}$$

Let

$$J(x,y,t) \equiv \sup_{(D,I)\in\mathcal{A}(x,y)} E_t\left[u(W_T)|x_t = x, y_t = y\right] \tag{8}$$

[6]Most results for the log utility case can be obtained by letting γ approach 1.

be the value function at time t. We summarize the main result for the no-transaction-cost case in the following theorem, with the notation convention that $t_{-1} = 0$.

Theorem 1 *Suppose that $\alpha = \theta = 0$. Then for $i = N, N-1, \ldots, 0$, the value function at time t is given by*

$$J(x, y, t) = \begin{cases} \frac{(x+y)^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\eta(t)} \left(\prod_{k=i+1}^{N} G_k^* \right), & t \in [t_{2i}, t_{2i+1}]; \\ \frac{(x+y)^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\eta(t)} \left(\prod_{k=i+1}^{N} G_k^* \right) G_i \left(\frac{y}{x+y}, t \right), & t \in (t_{2i-1}, t_{2i}); \end{cases} \tag{9}$$

and it is attained by the optimal trading policy of

$$\pi_t^* = \begin{cases} \pi_M(t), & t \in [t_{2i}, t_{2i+1}); \\ \pi_i^*, & t = t_{2i-1}, \end{cases}$$

when market is open, where

$$G_i(\pi, t) = E_t \left\{ \left[1 + \pi \left(R(t_{2i} - t) - 1 \right) \right]^{1-\gamma} \right\}, \tag{10}$$

$$R(h) = \exp\left[\left(\mu_n - r - \sigma_n^2/2 \right) h + \sigma_n \mathcal{B}(h) \right], \tag{11}$$

$$\pi_i^* = \arg\max_{\pi \in [0,1]} \frac{G_i(\pi, t_{2i-1})}{1-\gamma}, \quad G_i^* = G_i(\pi_i^*, t_{2i-1}), \tag{12}$$

and

$$\eta(t) = r(T - t) + \frac{(\mu_d - r)^2}{2\gamma\sigma_d^2} \sum_{i=0}^{N} (t_{2i+1} - t_{2i} \vee t)^+ \tag{13}$$

Without market closure, the optimal trading strategy is to invest a constant fraction of wealth in stock during daytime and a different fraction overnight because of the different return dynamics across day and night. With periodic market closure, Theorem 1 suggests that when market is open, the investor invests the same fraction of wealth in stock as in the case without market closure, but, facing market closure, the investor can no longer keep a constant fraction in stock during night. Instead, he adjusts his position at market close to a different fraction that is optimal on average, loosely speaking. In addition, since the investor cannot trade overnight, the stock position is stochastic overnight, can be suboptimal just before market open and therefore another discrete adjustment is also likely necessary at market open. These adjustments at market close and market open suggest that the trading volumes at these times are higher than during the rest of the trading hours, predicting a U-shaped trading volume pattern across trading hours, consistent with Hong and Wang (2000) and empirical evidence. Note that the optimal trading strategy during day is independent of parameter values during night. We show later that this is no longer true in the presence of transaction costs.

3. The transaction cost case

In the case where $\alpha + \theta > 0$, the problem is considerably more complicated. In this case, the investor's problem at time t becomes

$$V(x, y, t) \equiv \sup_{(D,I)\in\mathcal{A}(x,y)} E_t \left[u(W_T)|x_t = x, y_t = y\right]. \tag{14}$$

Under regularity conditions on the value function, for $i = N, N-1, ..., 0$, we have the following HJB equations for day time

$$\max\{V_t + \mathscr{L}V, (1-\alpha)V_x - V_y, -(1+\theta)V_x + V_y\} = 0, \forall t \in [t_{2i}, t_{2i+1}), \tag{15}$$

and for night time

$$V_t + \mathscr{L}V = 0, \forall t \in (t_{2i-1}, t_{2i}), \tag{16}$$

and at market close before T (i.e., $i \leq N-1$),

$$V(x, y, t_{2i+1}) = \max_{\Delta\in\mathcal{C}(x,y)} V(x - (1+\theta)\Delta^+ + (1-\alpha)\Delta^-, y + \Delta, t^+_{2i+1}), \tag{17}$$

with the terminal condition

$$V(x, y, T) = \frac{(x + (1-\alpha)y^+ - (1+\theta)y^-)^{1-\gamma}}{1-\gamma}, \tag{18}$$

where

$$\mathscr{L}V = \frac{1}{2}\sigma(t)^2 y^2 V_{yy} + \mu(t) y V_y + r x V_x,$$

and

$$\mathcal{C}(x,y) = \{\Delta \in \mathbb{R} : x - (1+\theta)\Delta^+ + (1-\alpha)\Delta^- \geq 0, y + \Delta \geq 0\}, \tag{19}$$

where the restriction set $\mathcal{C}$ imposes no borrowing or shorting overnight to ensure solvency.

As we show later, (15) implies that the solvency region for the stock

$$\mathcal{S} = \left\{(x,y) : x + (1-\alpha)y^+ - (1+\theta)y^- > 0\right\}$$

at each point in time during a day splits into a "Buy" region (BR), a "No-transaction" region (NTR), and a "Sell" region (SR), as in Davis and Norman (1990), Liu (2004), and Liu and Loewenstein (2002).

The following verification theorem shows the existence and the uniqueness of the optimal trading strategy. It also ensures the smoothness of the value function except in a set of measure zero.

Theorem 2 *(i) The value function is the unique viscosity solution of the HJB equation* (15)–(18).

(ii) The value function is $C^{2,2,1}$ *in* $(x,y) \in \mathcal{S} \setminus (\{y=0\} \cup \{x=0\})$, $t \in (t_{2i}, t_{2i+1})$ *and in* $x > 0$, $y > 0$, $t \in (t_{2i-1}, t_{2i})$, *for* $i = N, N-1, ..., 0$.

The homogeneity of the utility function u and the fact that $\mathcal{A}(\beta x, \beta y) = \beta \mathcal{A}(x,y)$ for all $\beta > 0$ imply that V is concave in (x,y) and homogeneous of degree $1-\gamma$ in (x,y) [cf. Fleming and Soner (1993), Lemma VIII.3.2]. This homogeneity implies that

$$V(x,y,t) = y^{1-\gamma}\varphi\left(\frac{x}{y}, t\right), \tag{20}$$

for some function $\varphi : (\alpha - 1, \infty) \times [0, T] \to \mathbb{R}$.[7]

Let

$$z = \frac{x}{y} \tag{21}$$

denote the ratio of the dollar amount invested in the bond to that in the stock. The homogeneity property then implies that Buy, No-transaction, and Sell regions can be described by two functions of time $z_b^*(t)$ and $z_s^*(t)$. The Buy region BR corresponds to $z_t \geq z_b^*(t)$, the Sell region SR to $z_t \leq z_s^*(t)$, and the No-Transaction region NTR to $z_s^*(t) < z_t < z_b^*(t)$. A time snapshot of these regions is depicted in Figure 1. As we show later, the optimal trading strategy in the stock when market is open is to transact a minimum amount to keep the ratio z_t in the optimal no-transaction region. Therefore the determination of the optimal trading strategy in the stock reduces to the determination of the optimal no-transaction region. At market close, because of the imminent market closure, the investor generally chooses a different no transaction region and the change in the boundaries implies possible lump sum trades at market close. For example, if the investor had a levered position just before market close, then a lump sum sale would be necessary to ensure overnight solvency. During market closure, the investor cannot trade and the ratio z_t fluctuates stochastically. Therefore, a lump sum trade may also be optimal at market open, because the risk exposure might have drifted away from the optimal one during the market closure period. In contrast to the no-transaction cost case, the optimal fraction of the wealth invested in the stock during daytime changes stochastically, since z_t varies stochastically due to no transaction in NTR.

The nonlinearity of the HJB equation and the time-varying nature of the free boundaries make it difficult to solve directly. Instead, we transform the above problem into a double obstacle problem, which is much easier to analyze.[8] All the analytical results in this paper are obtained through this

[7] Since the risk premium is positive, short sale is never optimal and thus $y > 0$.

[8] See Dai and Yi (2009) and the references therein for description of this class of problems and solution methodology.

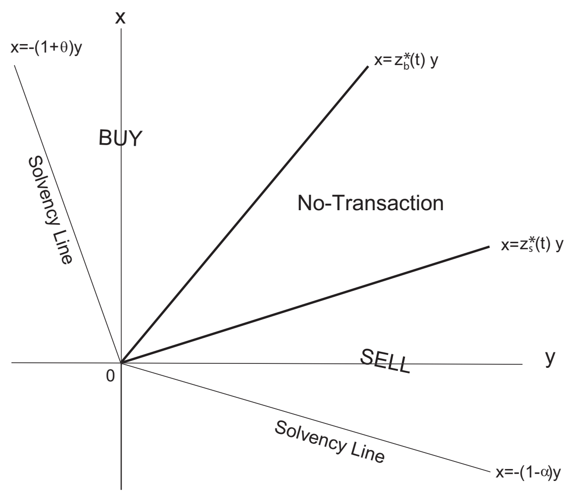

Figure 1: The Solvency Region

approach.

Let

$$z_M = \frac{\gamma \sigma_d^2}{\mu_d - r} - 1 \tag{22}$$

be the daytime Merton ratio (i.e., the optimal ratio in the absence of transaction costs). We then have the following comparative statics.

Proposition 1 *For any $t \in [t_{2i}, t_{2i+1})$, $i = N, N-1, ..., 0$, we have*

(i) $z_b^*(t) \geq (1+\theta) z_M$; *(ii)* $z_s^*(t) \leq (1-\alpha) z_M$.

Proposition 1 implies that if it is suboptimal to borrow or short sell in the absence of transaction costs (i.e., $z_M > 0$), then the no transaction region always brackets the Merton ratio.

Because the market closure time is deterministic and the investor can adjust his trading strategy accordingly, one might conjecture that the optimal buy and sell boundaries are always continuous in time from open to close (inclusive) so that transaction costs can be saved from discrete trades. The following proposition shows that this conjecture is incorrect.

Proposition 2 *The sell and buy boundaries have the following properties at t_{2i+1}, $i = N-1, N-2, ..., 0$:*

$$z_s^*\left(t_{2i+1}^-\right) = \min\left\{z_s^*(t_{2i+1}), (1-\alpha)\, z_M\right\}; \tag{23}$$

$$z_b^*\left(t_{2i+1}^-\right) = \max\left\{z_b^*(t_{2i+1}), (1+\theta)\, z_M\right\}. \tag{24}$$

As discussed above, when market closes, an investor adjusts his portfolio to be within the interval $[z_s^*(t_{2i+1}), z_b^*(t_{2i+1})]$. As confirmed by results in the next section, Proposition 2 suggests that an investor may optimally wait until the market closing time to discretely adjust his portfolio. For example, in the case $z_b^*\left(t_{2i+1}^-\right) = (1+\theta)z_M > z_b^*(t_{2i+1})$, if the investor's position is above the overnight buy boundary $z_b^*\left(t_{2i+1}\right)$ right before market closes, he will perform a discrete purchase to adjust his portfolio to $z_b^*(t_{2i+1})$. Similarly, an investor may make a discrete sale to adjust his portfolio to $z_s^*(t_{2i+1})$ if the position is below the overnight sale boundary $z_s^*(t_{2i+1})$ just before market close. Intuitively, given the much greater volatility during trading periods, eliminating all possible discrete trades at market close (i.e., choosing continuous trading boundaries) would likely require the investor make small but more frequent trades during trading hours, and thus would possibly incur even greater transaction costs than occasional discrete trades. Consistent with this intuition, as we show in the next section, the investor chooses no transaction regions such that lump sum trades at market close and market open occur only infrequently to avoid paying large transaction costs frequently.

By providing bounds on the boundaries, Propositions 1 and 2 also facilitate numerical computation of the boundaries.

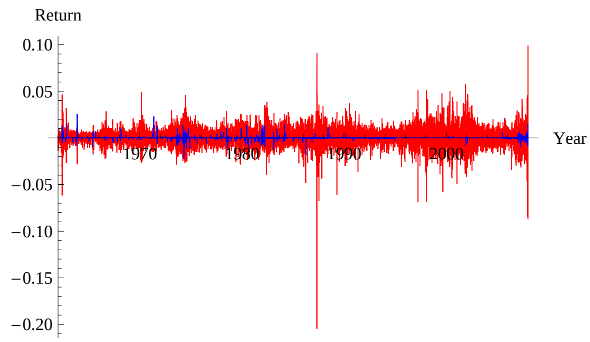

Figure 2: S&P 500 index returns
This figure plots the realized returns for S&P 500 index from January 1962 to October 2008, where the red path represents the simple return from market open to market close ("daytime" return) and the blue path represents the return from market close to next market open ("overnight" return).

4. Numerical analysis

In this section we provide numerical analysis of the impact of market closure and time-varying return dynamics on optimal trading strategy and liquidity premia, with the numerical procedure briefly described in Appendix A.5.

Figure 2 plots the realized returns for S&P 500 index from January 1962 to October 2008. This figure illustrates the much higher volatility during trading periods than that during nontrading periods, as shown in the literature.[9] Consistent with Figure 2, the existing literature on intraday price dynamics finds that the average per-hour ratio of day-time to overnight volatility is around 4.0 (e.g., Stoll and Whaley (1990), Lockwood and Linn (1990), Tsiakas (2008)). It is also found in the existing literature that expected returns are not significantly different across day and night. For example, on the comparison between the expected returns across day and night for six stock indices (including S&P 500, DJIA, NASDAQ 100), Tsiakas (2008) (p. 257) concludes that "Panel A indicates that among the six indexes, only for Paris are daytime and overnight expected returns

[9] Returns in this figure are not adjusted for duration difference between trading and nontrading periods. Such adjustment would make the volatility difference even more dramatic.

statistically different with 95% confidence."

Based on these findings, in the default case we set expected returns to be equal across day and night and use a lower volatility ratio value of $k = 3$, which biases against us in finding significant effects of market closure. To make the closest possible comparison with Constantinides (1986), we set default parameter values at $\mu_d = \mu_n = \mu = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\alpha = 0.5\%$, $\theta = 0.5\%$, $\gamma = 2$, and $T = 10$.[10] For simplicity, we assume that every day market opens for $\Delta t_d = 6.5$ hours (from 9:30am to 4pm) and closes for $\Delta t_n = 24 - 6.5 = 17.5$ hours.

Let the average (annualized) volatility be σ and the ratio of the day volatility to night volatility be $k \equiv \sigma_d/\sigma_n$. Then we have $\sigma_n = \sigma_d/k$, where

$$\sigma_d = k\sigma \cdot \sqrt{\frac{\Delta t_d + \Delta t_n}{k^2 \Delta t_d + \Delta t_n}}. \tag{25}$$

Then the volatility ratio of 3 implies that the volatility difference across day and night is equal to $\sigma_d - \sigma_n = 0.225$, with $\sigma_d = 0.337$ and $\sigma_n = 0.112$.

4.1. Optimal trading strategy

In Figure 3, we plot the initial optimal trading boundaries in terms of the fraction of wealth invested in the stock in the daytime and at the market close. Without transaction costs, the investor invests about 21.99% (Merton line) in the stock in the daytime and 100% at market close because of the higher overnight Sharpe ratio. Thus the investor buys at market close and sells at market open. Due to market closure, to avoid insolvency the investor cannot borrow or short at market close and invests between 56% (the red dot) and 100% (the blue circle) of his wealth in the stock at market close. In the day time the buy boundary is almost flat at 21.67%, very close to the daytime Merton

[10] Although both μ and r may be high relative to realizations in recent years, our numerical results demonstrate that keeping the risk premium constant, varying μ or r does not have significant impact for stock holding and so what matters is mainly the risk premium, as suggested by (7). In addition, our results are not sensitive to the choice of the investment horizon, as shown later.

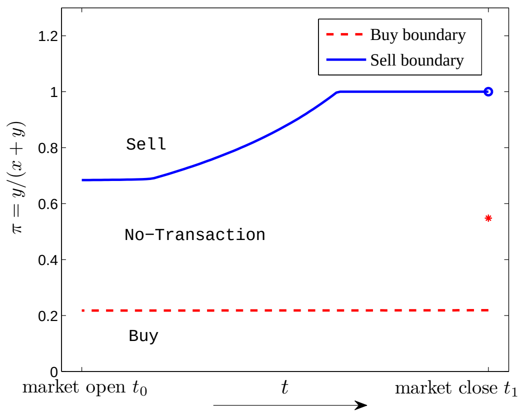

Figure 3: No-transaction regions across time.
Parameter default values: $T = 10$, $\gamma = 2$, $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\Delta_d = 6.5$ hours, $\Delta_n = 17.5$ hours, $\alpha = \theta = 0.005$, $\sigma_d = 0.337$, and $\sigma_n = 0.112$.

line. In the presence of transaction costs, however, the investor chooses a wide no-transaction region to reduce trading frequency. In particular, the sell boundary is well above the Merton line and increases to the sell boundary at market close (the blue circle). If just before market close, the position is below the buy boundary at market close (the red dot), then the investor buys to reach 56%. Due to no trade during market closure, the position just before next open may be outside the next daytime no-transaction region and thus may trigger another discrete trade at market open. The benefit of a large no transaction region is the reduction in transaction costs. The cost of this strategy is that in the daytime (resp. at market close) the investor holds significantly more (resp. less) in the stock than the optimal position for the no-transaction-cost case. This suggests that on average, the investor tries to smooth out market exposure across trading and nontrading periods due to transaction costs.

4.2. Liquidity Premia

Consider two perfectly correlated stocks with the same volatility, but one is subject to transaction cost and the other is not. For both stocks to be held in equilibrium, the expected return on the stock that is subject to transaction costs must exceed that of the liquid counterpart by some liquidity premium. Liquidity premia (defined by Constantinides (1986) as the maximum expected return an investor is willing to exchange for zero transaction cost) found by most theoretical portfolio selection models are well below empirical findings. For example, the seminal work of Constantinides (1986) finds that the liquidity premium to (round-trip) transaction cost (LPTC) ratio is only about 0.07 for a proportional (round-trip) transaction cost of 1%, while Amihud and Mendelson (1986) find that the LPTC ratio is about 1.9 for NYSE stocks. Using a regime switching model, Jang, Koo, Liu and Loewenstein (2007) show that the LPTC ratio given reasonable calibration is about 0.25. Lynch and Tan (2011) show that incorporating return predictability, state-dependent transaction costs and wealth shock can generate greater liquidity premia than Constantinides (1986). However, the liquidity premia found by Lynch and Tan (2011) with reasonable parameter values are still significantly smaller than the corresponding empirical evidence. In this section, we numerically demonstrate that if one takes into account periodic market closure and the resulting significant difference of volatilities across day and night, then transaction cost not only has a first order effect on liquidity premium, but the implied LPTC ratio can match empirical findings.

Let Market A be the actual market with positive transaction costs, different volatilities across day and night, and market closure at night. Let Market N be exactly the same as Market A except that there is no transaction cost during daytime in Market N. Let $V_A(x, y, t; \mu)$ and $V_N(x, y, t; \mu)$ be the time t value functions in these two markets respectively given the expected returns $\mu_d = \mu_n = \mu$.

Following Constantinides (1986), we solve

$$V_N(z_M, 1, 0; \mu - \delta) = V_A(z_M, 1, 0; \mu)$$

for the liquidity premium δ which measures how much an investor is willing to give up in the expected return to avoid transaction cost, when he starts at the daytime Merton ratio z_M. The liquidity premium δ is affected by the time varying volatility and the inability to trade overnight in Markets A and N. To separate out the effect of time varying volatility, we also compute the liquidity premium when an investor can trade with the same transaction costs day and night (i.e., no market closure) and thus leverage is allowed overnight. Specifically, let Market B be exactly the same as Market A except that the investor can trade overnight subject to the same daytime transaction costs and let Market M be exactly the same as Market B except that there are no transaction costs in Market M. We solve

$$V_M(z_M, 1, 0; \mu - \tilde{\delta}) = V_B(z_M, 1, 0; \mu)$$

for the liquidity premium $\tilde{\delta}$.[11]

In general, the effect of transaction cost on liquidity premium comes from two sources. One is the direct transaction cost payment incurred by trading. The other is the adoption of a trading strategy that would be suboptimal if there were no transaction cost. To understand which one is the main driving force behind the large increase in the liquidity premium, we also compute the liquidity premium caused by the suboptimal trading strategy alone. Specifically, let (I, D) be the optimal purchase and sale strategy in Market A and $V_N^{(I,D)}(x, y, 0; \mu)$ be the time 0 value function from following the strategy (I, D) in Market N (without transaction costs). We solve

$$V_N(z_M, 1, 0; \mu - \delta^0) = V_N^{(I,D)}(z_M, 1, 0; \mu)$$

[11] We have also computed the liquidity premia from comparing Market A to Market M. This alternative approach yields greater liquidity premia. For example, at $\alpha = \theta = 0.5\%$, the LPTC ratio is 1.84 and at $\alpha = \theta = 1\%$, it is 0.95.

for the liquidity premium δ^0 that is due to the adoption of a suboptimal trading strategy. For comparison, we compute the same measure for the model of Constantinides (1986).

In Table 1 we compare the liquidity premia, the LPTC ratios and the optimal no-transaction boundaries in this model with those reported by Constantinides (1986). This table suggests that liquidity premia significantly increase with transaction costs and are much higher in this model.[12] In fact, for a transaction cost rate of $< 1\%$ each way (e.g., for trading stock index such as S&P 500), transaction costs can have a first order effect. For example, at $\alpha = \theta = 0.5\%$, the LPTC ratio is as high as 1.76, more than 20 times higher than what is found by Constantinides (1986). This magnitude of LPTC ratio is consistent with empirical findings such as those by Amihud and Mendelson (1986) who find an LPTC ratio of 1.9. The second panel in Table 1 shows the results when the investor can trade overnight with the same transaction cost rate as in daytime. It suggests that if investors can also trade overnight the LPTC ratio becomes only slightly higher. Therefore, neither market closure *per se* nor the implied forced liquidation for levered daytime position is important for our results, what is important is the large volatility variation caused by market closure.[13]

One might suspect the greater liquidity premium may come from our assumption that an investor can trade *continuously* in the absence of transaction costs and thus the presence of transaction costs can significantly reduce the utility of the investor. To numerically demonstrate that our results are not driven by the "literal" continuous-time setting, we also compute the liquidity premium when we allow an investor to only trade at most twice a day with and without transaction costs.

[12] As the transaction costs increase, the difference between the two model decreases. This is because the investor optimally trades less often when transaction costs increase. Indeed, in the extreme case with $\alpha = 1$, in both models the investor never invests in stock and thus in both models the liquidity premia are equal to the risk premium of the stock (i.e., $\delta = \mu - r = 0.05$), which implies that there are no difference across these two models in terms of LPTC.

[13] The typical LPTC ratio found by Jang et. al is around 0.25. They report the ratio of liquidity premium to the one sided transaction costs (δ/α).

Table 1: Optimal Policy and Liquidity Premia against Transaction Cost Rates

$\alpha = \theta =$:	0.005	0.01	0.02	0.03	0.04	0.05	0.10	0.15
				This Model with Market Closure				
$z_b^*(0)$	3.590	3.608	3.644	3.680	3.718	3.753	3.932	4.189
$z_s^*(0)$	0.462	0.430	0.402	0.390	0.383	0.379	0.359	0.340
$z_b^*(t_1^-)$	3.567	3.585	3.621	3.656	3.692	3.727	3.905	4.089
$z_s^*(t_1^-)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$z_b^*(t_1)$	0.759	0.813	0.909	1.009	1.120	1.242	2.132	4.061
$z_s^*(t_1)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Liquidity Premium δ	1.76%	1.84%	1.96%	2.10%	2.24%	2.30%	2.80%	3.00%
$\delta/(\alpha+\theta)$	1.76	0.92	0.49	0.35	0.28	0.23	0.14	0.10
δ/δ_C	22.01	13.09	7.87	5.65	4.51	3.80	2.13	1.43
$\delta^0/\delta \times 100$	95.80	92.37	86.84	82.69	79.60	77.38	74.63	79.55
				This Model without Market Closure				
$\tilde{\delta}/(\alpha+\theta)$	1.84	0.95	0.50	0.36	0.28	0.23	0.14	0.11
$\tilde{\delta}/\delta_C$	23.01	13.56	8.05	5.73	4.54	3.82	2.14	1.44
				Constantinides (1986)				
$z_{b,C}^*$	0.690	0.726	0.783	0.832	0.877	0.920	1.122	1.326
$z_{s,C}^*$	0.566	0.561	0.555	0.550	0.546	0.542	0.525	0.509
Liquidity Premium δ_C	0.08%	0.14%	0.28%	0.36%	0.48%	0.60%	1.40%	2.10%
$\delta_C/(\alpha+\theta)$	0.08	0.07	0.07	0.06	0.06	0.06	0.07	0.07
$\delta_C^0/\delta_C \times 100$	9.50	13.79	20.44	24.38	27.49	30.08	35.79	36.10
				This Model, but with at most two trades a day				
$\tilde{\delta}/(\alpha+\theta)$	1.84	0.95	0.50	0.36	0.28	0.23	0.14	0.11
$\tilde{\delta}/\delta_C$	23.01	13.56	8.05	5.73	4.54	3.82	2.14	1.44

z_b^* and z_s^* are the buy and sell boundaries. t_1^- is just before first closing and t_1 is at first closing. δ, $\tilde{\delta}$, and δ_C are the time 0 liquidity premia, δ^0 measures the loss in risk premium from using the corresponding no trading region in the absence of transaction costs, all starting from the daytime Merton line. Other parameters: $\gamma = 2$, $T = 10$, $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\Delta t_d = 6.5$ hours, $\Delta t_n = 17.5$ hours, $\sigma_d = 0.337$, and $\sigma_n = 0.112$.

We find the results are almost identical, as reported in the last panel of Table 1.[14]

One typical explanation for a higher liquidity premium when investment opportunity set changes is the increase in transaction cost payment resulted from higher trading frequency (e.g., Jang et. al (2007)). To help understand whether higher transaction cost payment is also the main driving force behind the high LPTC ratio in our model, we also report the liquidity premium δ^0 that is solely due to the "suboptimal" trading strategy. In contrast to conventional wisdom, Table 1 shows that only a small percentage of the liquidity premium is from transaction cost payment. The vast majority of the liquidity premium comes from the "suboptimal" stock position. This finding suggests that with the large volatility difference, the investor chooses a wide no transaction region to reduce transaction cost payment at the cost of keeping significantly suboptimal average positions. Indeed, as Table 1 shows, the no-transaction region in this model is much wider than that in Constantinides (1986). For example, if $\alpha = \theta = 0.01$, the time 0 NTR in this model is $(0.430, 3.608)$ which is significantly wider than $(0.561, 0.726)$ that is optimal in Constantinides (1986).

However, wider no transaction regions do not necessarily imply the trading frequency in this model is lower than that in Constantinides (1986), because frequent market closure may increase rebalancing needs and thus also trading frequency. To compare the trading frequency and transaction cost payment across these two models, we conduct Monte Carlo simulations of 10,000 sample paths on these two models and report related results in Table 2.

Table 2 shows that the average time from a purchase to the next sale is about 2.8 years in our model in contrast to 1.8 years in Constantinides (1986). This suggests that the trading frequency

[14]In the default case, we assume market opens every day, ignoring the fact that market is closed during weekends and holidays. To see if this significantly biases our results, we also conduct the same analysis when we take into account the weekends and holiday closure. In addition, we also computed LPTC ratios for various risk aversion levels. We find the results are very similar, with LPTC ratios of 1.5 or higher. These results are not reported here to save space, but available upon request.

Table 2: Simulation Results

	This model		*Constantinides* (1986)	
$\alpha = \theta =$	0.005	0.01	0.005	0.01
Average daily \$ trading volume $\times 10^4$	3.8	3.2	1.3	1.2
PVTC	0.5%	0.9%	0.2%	0.4%
Average time from buy to sell (years)	2.8	4.5	1.8	2.6

PVTC is the discounted transaction costs paid as a percentage of the initial wealth. Other parameters: $\gamma = 2$, $T = 10$, $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\Delta t_d = 6.5$ hours, $\Delta t_n = 17.5$ hours, $\sigma_d = 0.337$, and $\sigma_n = 0.112$.

in our model is lower than that in Constantinides (1986), consistent with the fact that very few investors day trade even with significant Sharpe ratio variations across day and night. This confirms the intuition that to avoid large transaction cost payment, the investor chooses a trading strategy to significantly reduce trading frequency. On the other hand, Table 2 also shows that even though the trading frequency is lower, the transaction costs paid in this model are still greater than that in Constantinides (1986). For example, with 0.5% transaction cost rate, the present value of transaction costs paid is 0.5% of the initial wealth while it is only 0.2% in Constantinides (1986). This is mainly because trading in this model can involve large discrete trades at market close and market open, while in Constantinides (1986), only infinitesimal trading at the boundaries can happen after time 0. In other words the average per-trade trading size is greater in this model, which is also corroborated by the trading volume reported in Table 2.

In Figure 4, we plot the LPTC ratios against the day-night volatility difference $\sigma_d - \sigma_n$, holding the average volatility σ constant, for three different transaction cost levels of $\alpha = \theta = 0.5\%$, 0.75%, 1%. This figure shows that controlling for the same transaction cost, LPTC is sensitive to and increasing in the volatility difference across daytime and overnight. For example, at $\sigma_d - \sigma_n = 0.15$, the LPTC ratio is about 0.90 and it increases to 2.10 when $\sigma_d - \sigma_n$ increases to 0.25. It is worth

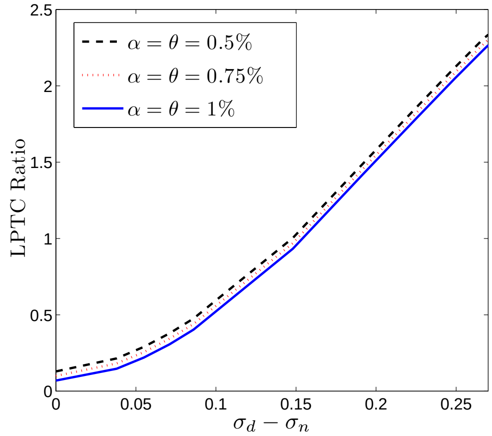

Figure 4: LPTC ratios against day-night volatility difference.
Parameter default values: $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\Delta t_d = 6.5$ hours, and $\Delta t_n = 17.5$ hours.

noting that at $\sigma_d - \sigma_n = 0$, the LPTC ratio is close to that of Constantinides (1986). This suggests that the effect of the presence of intertemporal consumption on liquidity premium is small. In addition, Figure 4 also suggests that controlling for the same volatility difference, the LPTC ratio decreases with transaction costs, consistent with Table 1. These results motivate some of our subsequent empirical analysis.

One concern about our results may be that we have assumed that the expected returns across day and night be the same. This assumption is motivated by the empirical findings that either the expected returns do not vary significantly across trading and nontrading periods or the returns over the nontrading periods are significantly *higher* than those over the trading periods (e.g., Tsiakas (2008)). While it is beyond the scope of this paper to provide an equilibrium model that can generate different Sharpe ratios across day and night, such an equilibrium can be consistent with an economy with heterogeneous investors, e.g., some investors may be more risk averse toward carrying overnight inventories than others, or they may have heterogeneous beliefs on the time

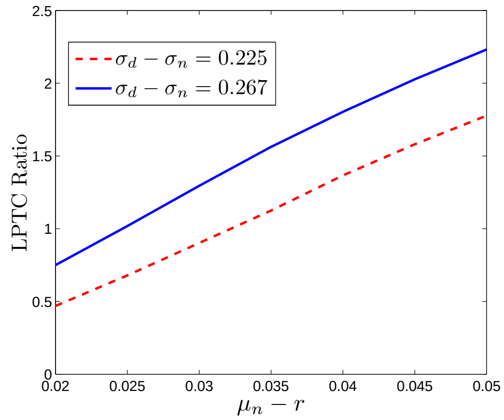

Figure 5: LPTC ratios against overnight risk premium $\mu_n - r$.
Parameter default values: $\mu_d = 0.15$, $r = 0.10$, $\alpha = \theta = 0.005$, $\Delta t_d = 6.5$ hours, $\Delta t_n = 17.5$ hours, $\sigma_d = 0.337$, and $\sigma_n = 0.112$.

varying return dynamics. As we have shown above, because of the presence of transaction costs, investors trade infrequently even when the overnight Sharpe ratio is much greater. This suggests that with transaction costs, small heterogeneity may be sufficient to sustain an equilibrium with different Sharpe ratios across trading and nontrading periods.

However, as a robustness check, we present Figure 5 to show how LPTC varies as a function of the overnight risk premium. Figure 5 shows that the LPTC ratio increases with the overnight expected return and even if the night time expected return is significantly lower than that in the day time, the LPTC ratio can still be greater than 1. For example, suppose the overnight risk premium is only 3.5%, 30% lower than that in the day time (5%). Figure 5 indicates that for volatility difference of 0.225, the LPTC ratio is still as high as 1.05, while for volatility difference of 0.267 (corresponding to a volatility ratio of 4), the LPTC ratio becomes 1.52. This is because the overnight volatility is much smaller than the daytime volatility.

Another concern about the high LPTC ratio may be that in the main model there is only one

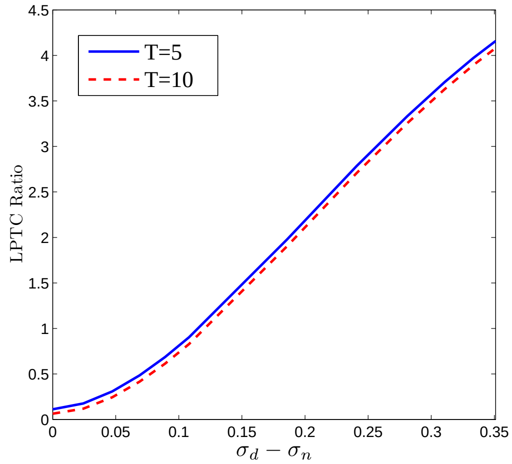

Figure 6: LPTC ratios against day-night volatility difference: the Two-stock Case.
Parameter default values: $\mu_L = 0.03$, $\mu_I = 0.08$, $\sigma_L = 0.15$, $\sigma_I = 0.25$, $\rho = 0.7$, $k_L = 1$, $\Delta t_d = 6.5$ hours, $\Delta t_n = 17.5$ hours, and $\alpha = \theta = 0.005$.

stock the investor can trade. If a less illiquid and correlated stock were available, then the investor would be able to trade more in the less illiquid stock to achieve a similar risk exposure at a lower cost and thus the LPTC ratio for the illiquid stock would be lowered. To examine this possibility, in Figure 6 we plot the LPTC ratio against the day-night volatility difference for the illiquid stock from solving a two-stock model.[15] Figure 6 shows that our results are robust to the availability of a liquid (zero transaction cost) and highly correlated stock ($\rho = 0.7$) and the LPTC ratio still monotonically increases with the volatility difference. In addition, Figure 6 suggests that the LPTC ratio can be quite insensitive to a change in the investment horizon. For example, shortening the investment horizon by 5 years only slightly increases the ratio.

[15]This two-stock model is presented in an earlier version of the paper. To save space, we omit it in this version. The subscripts L and I for default parameter values denote the liquid and illiquid stocks respectively.

5. A three-period extension

In our main model, we assume that either market is open and an investor can trade subject to relatively small transaction costs or market is closed and the investor cannot trade at all. However, in practice, some investors can trade in the after-hour market although at higher transaction costs. We next consider a three-period extension of our main model to examine the impact of this additional trading opportunity on our main results.

More specifically, the investment horizon T is partitioned into $0 = t_0 < ... < ... < t_{3N+1} = T$. Time intervals $[t_{3i}, t_{3i+1}]$ denote regular-hour trading periods within which an investor can trade as in the day period in the main model, time intervals $(t_{3i+1}, t_{3i+2}]$ represent after-hour trading periods within which the investor can also trade the stock but at higher transaction cost rates α_a and θ_a, and the market is closed and thus no trading takes place in $(t_{3i+2}, t_{3i+3}), \forall i = 0, 1, ..., N$. The stock price evolves as in (1) with

$$\mu(t) = \begin{cases} \mu_d, & \text{regular-hour} \\ \mu_a, & \text{after-hour} \\ \mu_n, & \text{night} \end{cases} \quad \text{and} \quad \sigma(t) = \begin{cases} \sigma_d, & \text{regular-hour} \\ \sigma_a, & \text{after-hour} \\ \sigma_n, & \text{night,} \end{cases}$$

where $\mu_d > r, \mu_a > r, \mu_n > r, \sigma_d > 0, \sigma_a > 0, \sigma_n > 0$ are assumed to be constants. The dynamics for the dollar amount x_t in the risk free asset and the dollar amount y_t in the stock are the same as in (2) and (3) except for the after-hour period, α and θ are replaced by α_a and θ_a respectively.

Let $V(x, y, t)$ be the value function in this three-period model similarly defined as in (14). Similar to the main model, we have the HJB equations for regular-hour trading period:

$$\max\{V_t + \mathscr{L}V, (1-\alpha)V_x - V_y, -(1+\theta)V_x + V_y\} = 0, \quad t \in [t_{3i}, t_{3i+1}],$$

for the after-hour trading period:

$$\max\{V_t + \mathscr{L}V, (1-\alpha_a)V_x - V_y, -(1+\theta_a)V_x + V_y\} = 0, \quad t \in (t_{3i+1}, t_{3i+2}],$$

and for the no-trading period:

$$V_t + \mathscr{L}V = 0, \quad t \in (t_{3i+2}, t_{3i+3}),$$

where $\mathscr{L}V$ is as defined in (19). For connecting conditions between periods, we have at the regular-hour ending time $t = t_{3i+1}$,

$$V(x, y, t_{3i+1}) = \max_{\Delta \in \mathcal{C}(x,y)} V\left(x - (1+\theta)\,\Delta^+ + (1-\alpha)\Delta^-, y + \Delta, t_{3i+1}^+\right), \tag{26}$$

and at the market closing time $t = t_{3i+2}$,

$$V(x, y, t_{3i+2}) = \max_{\Delta \in \mathcal{C}_a(x,y)} V\left(x - (1+\theta_a)\,\Delta^+ + (1-\alpha_a)\Delta^-, y + \Delta, t_{3i+2}^+\right), \tag{27}$$

where $\mathcal{C}(x, y)$ is as in (19) and

$$\mathcal{C}_a(x, y) = \{\Delta \in \mathbb{R} : x - (1+\theta_a)\Delta^+ + (1-\alpha_a)\Delta^- \geq 0, y + \Delta \geq 0\}. \tag{28}$$

The terminal condition at T is the same as in (18). A verification theorem can be proven using similar arguments to that for Theorem 2. As before, using homogeneity we can write the value function in the form as in (20).

To facilitate numerical computation of the boundaries, we also derive similar results to Propositions 1 and 2.[16] Let z_M be as defined in (22) and similarly we define

$$z_M^a = \frac{\gamma \sigma_a^2}{\mu_a - r} - 1. \tag{29}$$

Proposition 3 *For $i = N, N-1, ..., 0,$ we have*

(i) $z_b^*(t) \geq (1+\theta_d) z_M$; $z_s^*(t) \leq (1-\alpha_d) z_M$, for any $t \in (t_{3i}, t_{3i+1})$.

(ii) $z_b^*(t) \geq (1+\theta_n) z_M^a$; $z_s^*(t) \leq (1-\alpha_a) z_M^a$, for any $t \in (t_{3i+1}, t_{3i+2})$.

[16]The proofs of Propositions 3 and 4 are similar to those for Propositions 1 and 2 and thus omitted to save space.

Proposition 4 *The sell and buy boundaries have the following properties for $i = N, N-1, ..., 0:$*

(i) at t_{3i+1},

$$\begin{aligned} z_s^*(t_{3i+1}^-) &= \min\{z_s^*(t_{3i+1}), (1-\alpha)z_M\}; \\ z_b^*(t_{3i+1}^-) &= \max\{z_b^*(t_{3i+1}), (1+\theta)z_M\}; \end{aligned}$$

(ii) at t_{3i+2},

$$\begin{aligned} z_s^*(t_{3i+2}^-) &= \min\{z_s^*(t_{3i+2}), (1-\alpha_a)z_M^a\}; \\ z_b^*(t_{3i+2}^-) &= \max\{z_b^*(t_{3i+2}), (1+\theta_a)z_M^a\}. \end{aligned}$$

We use a similar numerical procedure to that for the main model to solve for the optimal trading strategy and liquidity premium. We report the results in Table 3. In the base case, we set the after-hour trading period Δ_{t_a} to be 3 hours, with the rest of the parameter values remaining the same as those in Table 1. Table 3 shows that not only qualitatively results are the same as in the main model, the magnitudes are also similar. For example, the buy boundaries just before and at the end of the regular-trading period are close to those in Table 1, from 3.567 to 3.575 and 0.759 to 0.824 respectively. In addition, the liquidity premium magnitudes are only slightly lower than those in the two period model, 1.75 versus 1.76. We find that changing the length of the after-hour trading period or the transaction costs in the after-hour trading period does not have significant impact on these results. Overall, the effect of after-hour trading seems relatively small. As shown before, the main driving force behind the higher liquidity premia in our model is the high opportunity cost of not being able to rebalance costlessly to take advantage of the time varying volatility. While adding the extended trading hours helps investor better manage the after-hour portfolio, it does not significantly reduce the rebalancing cost from the optimal regular-hour position to the optimal after-hour position in the absence of transaction cost. In addition, because with even very small transaction cost an investor would trade already very infrequently, the reduction of after-hour transaction cost from infinity (i.e., the two period model, no after-hour trading) to a small positive level (e.g., 2.5%, very infrequent trading) does not significantly change the value function. Therefore, adding the extended trading hours only changes the numerical results slightly.

Table 3: Optimal Policy and Liquidity Premia against Transaction Cost Rates

	$z_b^*(t_1^-)$	$z_b^*(t_1)$	$z_b^*(t_2^-)$	$z_b^*(t_2)$	$\delta/(\alpha+\theta)$	δ/δ_C	δ_0/δ
Base Case	3.575	0.824	0.820	0.799	1.75	23.20	96.98
$\alpha_a = \theta_a = 0.05$	3.575	0.846	0.840	0.799	1.75	23.20	96.09
$\alpha_a = \theta_a = 0.10$	3.575	0.885	0.804	0.799	1.75	23.20	94.67
$\alpha_a = \theta_a = 0.15$	3.575	0.925	0.920	0.799	1.75	23.20	92.73
$\Delta_{t_a} = 5$	3.575	0.931	0.926	0.902	1.75	23.08	97.00
$\Delta_{t_a} = 10$	3.575	1.459	1.445	1.407	1.75	23.96	97.41
$k = 4$	4.097	0.837	0.837	0.814	2.21	31.18	98.38

Base case: $\gamma = 2$, $\mu_d = \mu_a = \mu_n = 0.15$, $r = 0.10$, $\alpha = \theta = 0.005$, $\alpha_a = \theta_a = 0.025$, $\Delta_{t_d} = 6.5$, $\Delta_{t_a} = 3$, $\Delta_{t_n} = 24 - \Delta_{t_d} - \Delta_{t_a}$, $\sigma_d = 0.337$, $\sigma_a = \sigma_n = 0.112$, and $T = 10$.

6. Empirical analysis of the impact of the volatility difference on liquidity premia

There are three main conclusions from the above analysis: (1) Market closure and the volatility difference between the trading and non-trading periods may change the optimal trading strategy significantly; (2) The transaction costs can have a large impact on liquidity premia comparable to empirical findings; and (3) Conditional on the same increase in the transaction costs (e.g., from 0 to 1%), stocks with greater volatility variation across trading and nontrading periods require higher additional liquidity premia, as shown in Figures 4 and 5.

There is a vast literature on the determinants of liquidity premia (e.g., Amihud and Mendelson (1986) and Eleswarapu (1997)). However, as far as we know, our model is the only one that suggests volatility variation across trading and nontrading periods can be an additional determinant of liquidity premia. In addition, no empirical studies have investigated such a possibility. Therefore, in this section, we empirically investigate this third main result (i.e., whether indeed the volatility variation significantly affects liquidity premia) by closely following the methodology of Eleswarapu (1997).

6.1. Data and portfolio formation

As argued by Eleswarapu (1997), bid-ask spreads for NASDAQ stocks better represent the cost of transacting than those for NYSE stocks. Accordingly, we perform our analysis on NASDAQ stocks and use relative bid-ask spread to measure transaction costs as Eleswarapu (1997). Because of the limited availability of daily open and closing prices, we use the sample period of 1991-2012.[17] The primary data consists of daily open and closing prices and closing bid and ask prices of Nasdaq stocks provided by the Center for Research in Security Prices (CRSP). Since there are 6.5 trading hours in a normal trading day and the hours between close and next open may vary, the continuously compounded daily returns for the trading period (r_{dt}) and for the non-trading period (r_{nt}) for stock i are computed as

$$r^i_{dt} = \frac{24}{6.5} \log \left(\frac{\textit{closing price}}{\textit{open price}} \right), \tag{30}$$

$$r^i_{nt} = \frac{24}{\textit{hours between open and previous close}} \log \left(\frac{\textit{open price}}{\textit{previous closing price}} \right). \tag{31}$$

This takes into account weekends and holidays market closures. For stocks with cash dividends, stock splits, and stock dividends events, we use the CRSP daily events distribution database to make corresponding adjustments in return calculations.

For each stock, the spread in a month is calculated by averaging the daily relative bid-ask spread in the month, where the relative spread is equal to the dollar closing bid-ask spread divided by the

[17] In an earlier version, we did our analysis using the sample period of 1991-2007 to address the possible concerns over the abnormal effects of the financial crisis. The qualitative results are not affected by the choice of the sample period.

closing mid-quote price, *i.e.*,

$$Spread_{it} = \frac{1}{N_{it}} \sum_{1}^{N_{it}} \frac{closing\ ask - closing\ bid}{(closing\ ask + closing\ bid)/2}, \tag{32}$$

where N_{it} is the number of trading days in month t for stock i.

For our test, we form 245 equally-weighted portfolios from triple-sorting by average volatility difference across trading and nontrading periods in the previous year (5 groups), average relative bid-ask spreads in the previous year (7 groups), and their estimated betas in the last three years (7 groups). The volatility difference across trading and nontrading periods for a month is computed using the daily returns r_{dt} and r_{nt} in the month. We assume that the continuously compounded returns in the trading and nontrading periods are normally distributed with constant means and variances. Because the observed nontrading periods returns r_{nt}^{i} may be unevenly spaced across time, we use the weighted (by the square-root of time between observations) least square regression method to estimate the volatilities to address the potential heteroskedasticity problem.

Betas of individual stocks are estimated using market model regressions with data over the three-year portfolio formation period prior to the test year:

$$r_{it} = \alpha_i + \beta_i r_{mt} + \varepsilon_{it}, t = 1, 2, ..., 36, \tag{33}$$

where r_{it} and r_{mt} are the month t excess returns (over the corresponding one-month Treasury bill return) on stock i and on the market index, respectively. We use the value-weighted portfolio of all NASDAQ stocks as the market index.

For each test year, stocks are ranked and divided into five groups as evenly as possible based on the average volatility difference in the previous year. Each of these five groups is then divided into seven equal subgroups according to their average bid-ask spread in the previous year. Finally, each of the subgroups is ranked and divided into seven equal sub-subgroups according to their estimated beta coefficients for the previous three-year period. Therefore, there are 245 test portfolios with approximately equal number of stocks. The monthly portfolio returns in a test year are computed by averaging the excess returns of the stocks in each of the 245 portfolios each month. This portfolio formation procedure is performed for each of the 19 test years (1994-2012).

6.2. Descriptive statistics

6.2.1. Trading versus non-trading return volatility

Table 4 shows that the average annualized volatility difference is 0.70 for all NASDAQ stocks during the period 1993-2012, ranging from 0.12 for the lowest quintile to 1.51 for the highest quintile.[18]

To compare our volatility variation results with those in the existing literature, we also compute the per-hour ratios of return volatilities in trading versus nontrading periods and report them in Table 5. Table 5 shows that the average per-hour volatility ratio of trading versus non-trading periods is 3.24, ranging from 2.10 for the lowest trading volume quintile to 4.07 for the highest quintile.[19] Consistent with our model's implication and the results in Stoll and Whaley (1990), Table 5 also suggests that as volatility ratio increases, the daily dollar trading volume of a stock increases.

6.2.2. Average spread, Beta, and market value of equity for the portfolios

In Table 6, we report average spread, beta, and market value of equity for the 49 spread-beta sorted portfolios (formed from the 245 test portfolios by pooling the portfolios with the same ranks according to spread and beta) over the 19 test-year periods. Table 6 shows that portfolio spreads range from 0.780% to 11.136% for the time period 1993-2012. In contrast, the portfolio spreads range from 1.87% to 32.53% for the time period 1976-1990 as reported in Eleswarapu (1997). Thus,

[18] The annualization is done to be consistent with what we use in previous sections.

[19] The dollar trading volume of a stock is calculated as the closing price multiplied by the total number of shares of a stock sold on each day.

Table 4: Average Volatility-difference for NASDAQ Stocks during the Period 1993-2012

	All stocks	By volatility-difference annual average quintile[a]				
		Smallest	2	3	4	Largest
Average volatility-difference	0.0445	0.0076	0.0257	0.0387	0.0548	0.0956
Standard error [b] ($\times 10^{-2}$)	0.0525	0.0438	0.0084	0.0088	0.0112	0.033
Average number of firms	3664	732	733	733	733	733

Average volatility-difference is calculated as follows: (1) the difference between open-to-close return volatility and previous close-to-open return volatility is calculated for each stock for each month, and then averaged for each year; (2) the annual average is then averaged across all stocks in the sample and all stocks in each quintile; (3) the resulting average is then averaged across 20 years.

[a]Stocks are ranked by volatility-difference annual average and divided into quintiles each year.

[b]The standard error is based on the distribution of the 20 yearly average of volatility-difference annual average.

Table 5: Average Ratios of Open-to-close Return (r_{dt}) Volatility to Previous Close-to-open Return (r_{nt}) Volatility for NASDAQ Stocks during the Sample Period 1993-2012

	All stock	By daily dollar volume quintile[a]				
		Smallest	2	3	4	Largest
Average ratio	3.24	2.10	2.73	3.44	3.86	4.07
Standard error[b]	0.025	0.011	0.025	0.036	0.033	0.028
Average number of firms	3664	732	733	733	733	733
Sample size	240	240	240	240	240	240
Average dollar volume($\times$1000)	11361.67	24.72	125.32	511.87	2134.17	54028.33

Volatility ratios are calculated for each stock in each month and are then averaged across all stocks in the sample and all stocks within each dollar trading volume quintile. Finally, the average monthly ratios are averaged across the 240 months in the 20-year sample period.

[a]Stocks are ranked within each month by average daily dollar trading volume in the month and divided into quintiles each month during the sample period. The dollar trading volume of a stock is calculated using the closing price × the total number of shares of a stock sold on each day.

[b]The standard error is based on the distribution of the 240 monthly average ratios.

Table 6: Average Relative Bid-Ask Spread, Betas and Size (Market Value of Equity) for the 49 Spread/Beta Portfolios of NASDAQ Firms, 1994-2012

Assignment of a stock to a particular spread/beta portfolio in a given test year depends on two criteria: 1) the average spread in the previous year (7 groups), and 2) a stock's beta estimated with 36 months of preceding returns (7 groups). In this table, we report the average spread, betas and market value of equity for the 49 portfolios. Each cell contains three entries. The top number is the relative bid-ask spread of the portfolio. The portfolio spread is the average spread of the stocks in the portfolio in the year preceding the test year. The second number is the estimated portfolio beta computed with 228 months of portfolio return data (1994-2012). The third number is the market value of equity (size), where the equity value of the firm is computed in December in the year preceding the test year. Portfolio spreads and market value of equity are averaged over the 19 years, 1994-2012.

	Spread (in percentage), Beta, Size (in millions)							
	Beta group							
Spread Group	Lowest	2	3	4	5	6	Highest	Mean
Lowest	0.780	0.752	0.723	0.706	0.693	0.696	0.705	0.722
	1.032	1.065	1.194	1.252	1.276	1.401	1.657	1.268
	2561	2358	3172	3490	4531	6032	5665	3973
2	1.355	1.348	1.338	1.330	1.323	1.332	1.326	1.336
	1.069	1.111	1.146	1.203	1.295	1.438	1.582	1.263
	628	556	630	558	610	853	910	678
3	1.918	1.901	1.884	1.908	1.883	1.894	1.885	1.896
	1.020	1.053	1.150	1.177	1.314	1.366	1.578	1.237
	286	285	295	288	330	355	584	346
4	2.593	2.588	2.583	2.570	2.583	2.587	2.565	2.581
	1.038	1.059	1.144	1.168	1.239	1.375	1.563	1.227
	175	167	168	182	170	201	273	191
5	3.552	3.547	3.576	3.530	3.549	3.531	3.510	3.542
	1.053	1.072	1.073	1.210	1.215	1.250	1.488	1.194
	102	129	107	98	104	93	143	111
6	5.135	5.109	5.108	5.099	5.045	5.062	5.051	5.087
	0.938	0.973	1.026	1.041	1.204	1.222	1.469	1.125
	55	54	53	59	58	56	57	56
Highest	10.155	11.136	9.615	9.780	9.605	9.618	10.281	10.027
	0.847	0.893	0.930	0.934	0.989	1.144	1.306	1.006
	30	27	26	26	25	27	30	28
Mean	3.785	3.629	3.566	3.540	3.523	3.531	3.617	3.599
	1.009	1.030	1.088	1.141	1.218	1.314	1.521	1.189
	549	507	639	672	832	1088	1095	769

the spreads have become significantly smaller in recent years. The portfolio betas range from 0.847 to 1.657. The portfolio market value is computed by averaging the market value of equity (size) of the firms in the portfolio in the December preceding a test year. The average equity values range from \$ 25 million to \$ 6.03 billion. Consistent with Eleswarapu (1997), we find that stocks with a higher market equity value tend to have smaller spreads as illustrated in Table 6.

6.3. Empirical results

We examine the cross-sectional relation between excess return and spread for the 98 portfolios in the lowest and the highest quintiles of the volatility-difference groups using the Fama and Macbeth type regressions as in Eleswarapu and Reinganum (1993), and Eleswarapu (1997). The excess returns of the 98 portfolios are regressed on their unconditional betas, spreads, and Log(Size) each month.[20] To test the impact of volatility-difference across trading and nontrading periods on the relation between excess return and spread, we interact spread with a dummy variable (Dummy) that is set to 1 for the highest quintile and 0 for the lowest one and include this interaction term as an additional variable in the regression. The time-series average of the monthly regression coefficients and the corresponding standard errors are reported in Table 7. Similar to Amihud and Mendelson (1986) and Eleswarapu (1997), the first two columns of Table 7 show that transaction costs significantly affect excess returns. Indeed, the highly significant coefficient of spread implies a 1% increase in the spread is associated with a 0.22% increase in the monthly risk-adjusted excess return. However, the third column of Table 7 implies that this significant impact of transaction costs mainly comes from stocks with high volatility-differences across trading and nontrading periods. More specifically, the coefficient of the interaction term $Spread \times Dummy$ is large, positive, and statistically significant at the 0.1% level, while the coefficient of spread is no longer significantly different from 0. These

[20] The average spreads and sizes in the previous year are used in the regressions. Similar to Fama and French (1992), we use the unconditional portfolio betas estimated using the monthly portfolio returns from all of the test-year periods.

results imply that as our model suggests, stocks with greater volatility-differences require higher additional liquidity premium for the same increase in the transaction costs. For example, for a 1% increase in the spread, stocks with high volatility-differences require 0.36% higher monthly risk adjusted excess return than those with low volatility-differences.

An alternative explanation for the above result may be as follows. Small firms are typically less frequently traded and so liquidity premia per unit of transaction costs might be higher for small firms than large firms. In the meantime, small firm returns are also typically more volatile which might drive the volatility difference across trading and nontrading periods higher than that for other stocks. Therefore, liquidity premia per unit of transaction costs are higher for stocks with large volatility differences might be because both liquidity premia per unit of transaction costs and volatility differences are typically higher for small firms than large firms. To address this concern, we include this interaction term of $Spread$ with $Log(size)$ in the regression and report the result in the fourth column. Column 4 shows that the coefficient for the interaction term $Spread \times Dummy$ is still positive and significant after controlling for this size effect. Consistent with this finding, Column 2 indicates that after controlling for spread, firm size is not statistically significant in affecting liquidity premia.

We have also done robustness checks such as controlling for book-to-market effect, trading volume effect, Fama and French three factors and Carhart (1997) four factors, the results remain essentially the same.[21] Overall, our results strongly support our prediction that the volatility difference across trading and non-trading periods is an important determinant of liquidity premia, as suggested by our analysis.

[21] These results are reported in an earlier version of the paper.

Table 7: Fama-MacBeth Type Regressions for the 98 Portfolios of the Lowest Quintile and Highest Quintile, based on Volatility-Difference of NASDAQ Firms, 1994-2012

$$(A):\ Return_{pt} = a_0 + a_1\, \beta eta_{pt} + a_2 Spread_{pt} + e_{pt}$$

$$(B):\ Return_{pt} = b_0 + b_1\, \beta eta_{pt} + b_2 Spread_{pt} + b_3 Log(Size_{pt}) + e_{pt}$$

$$(C):\ Return_{pt} = c_0 + c_1\, \beta eta_{pt} + c_2 Spread_{pt} + c_3 Log(Size_{pt}) + c_4\, Dummy_{pt} + c_5\, Spread_{pt} \times Dummy_{pt} + e_{pt}$$

Assignment of a stock to a particular variance-difference/spread/beta portfolio in a given test year depends on three criteria: 1) the average difference between the trading and non-trading return variances in the previous year, 2) the average spread in the previous year, and 3) a stock's beta estimated with 36 months of preceding returns. In the cross-sectional regression, the portfolio spread (*Spread*$_{pt}$) is computed from the average of the firm's spread in the preceding year. The Size$_{pt}$ (equity value) is the value in December in the year preceding each test year. The portfolio beta (βeta_{pt}) is the unconditional beta which is computed using the monthly portfolio returns from all the test period years. The dummy variable (Dummy$_{pt}$) is 1 for portfolios in the highest quintile and 0 for portfolios in the lowest quintile, based on the difference between the trading and non-trading period return variances. The cross-sectional regression is fit in each month, *t*, of the test-period years. The coefficients are the time-series (168 months) means with corresponding standard errors in parentheses.

	Regression		
Variable	(A)	(B)	(C)
βeta	-0.0026 (0.0058)	-0.0022 (0.0058)	0.0128 (0.0066)
$Spread$	0.2784*** (0.0560)	0.2718*** (0.0485)	-0.1043 (0.0598)
$Log(Size)$		0.0002 (0.0010)	-0.0019 (0.0011)
$Dummy$			-0.0380*** (0.0071)
$Spread \times Dummy$			0.4460*** (0.0793)
N	168	168	168

*** Statistically significant at the 0.1% level.

7. Concluding remarks

Existing portfolio choice literature ignores periodic market closure and the significantly different volatilities across trading and nontrading periods. Therefore, the optimal trading strategy that is relevant for practice is still largely unknown. In this article, we show that incorporating periodic market closure and the return dynamics across trading and nontrading periods leads to significantly different trading strategy. In addition, we numerically demonstrate that transaction costs can have a first order effect on liquidity premia that is largely comparable to empirical findings. Furthermore, we provide empirical support for the importance of the volatility difference across trading and nontrading periods in affecting liquidity premia. As far as we know, this is the first paper that finds volatility variation across trading and nontrading periods is an important determinant of liquidity premia.

APPENDIX

A.1 Proof of Theorem 1

To begin with, we point out

$$\sum_{k=0}^{N}\left(t_{2k+1}-t_{2k}\vee t\right)^{+}=\begin{cases}\sum_{k=i+1}^{N}\left(t_{2k+1}-t_{2k}\right)+t_{2i+1}-t, & \text{if } t\in\left[t_{2i},\,t_{2i+1}\right),\\ \sum_{k=i}^{N}\left(t_{2k+1}-t_{2k}\right), & \text{if } t\in\left[t_{2i-1},\,t_{2i}\right),\end{cases} \tag{A-1}$$

which means the cumulative time in day.

When $t\in[t_{2N},\,t_{2N+1})$, the theorem is the well-known Merton's result, where we follow the Merton's strategy π_M.

When $t\in(t_{2N-1},\,t_{2N})$, no trading is allowed, then

$$\begin{aligned}J\left(x,\,y,\,t\right) &= E_t^{x,\,y}\left[J(x_{2N},y_{2N},t_{2N})\right]\\ &= \frac{1}{1-\gamma}E_t^{x,\,y}\left[\left(x_{t_{2N}}+y_{t_{2N}}\right)^{1-\gamma}\right]e^{(1-\gamma)\eta(t_{2N})}.\end{aligned} \tag{A-2}$$

It is easy to verify that

$$E_t^{x,\,y}\left[\left(x_{t_{2N}}+y_{t_{2N}}\right)^{1-\gamma}\right]=\left(x+y\right)^{1-\gamma}e^{(1-\gamma)r(t_{2N}-t)}G_N\left(\frac{y}{x+y},\,t\right).$$

Substituting into (A-2), we then get

$$J\left(x,\,y,\,t\right)=\frac{1}{1-\gamma}\left(x+y\right)^{1-\gamma}e^{(1-\gamma)\eta(t)}G_N\left(\frac{y}{x+y},\,t\right), \tag{A-3}$$

where we have used $\eta\left(t_{2N}\right)+r\left(t_{2N}-t\right)=\eta\left(t\right)$ due to (13) and (A-1).

When $t=t_{2N-1}$ at which trading is allowed, we need to determine the optimal strategy $\pi\in[0,1]$. Due to (A-3), we get

$$\begin{aligned}J\left(x,\,y,\,t_{2N-1}\right) &= \sup_{\pi\in[0,1]}\frac{1}{1-\gamma}\left(x+y\right)^{1-\gamma}e^{(1-\gamma)\eta(t_{2N-1})}G_N\left(\pi,\,t_{2N-1}\right)\\ &= \frac{1}{1-\gamma}\left(x+y\right)^{1-\gamma}e^{(1-\gamma)\eta(t_{2N-1})}G_N^{*},\end{aligned}$$

where we have chosen the optimal strategy

$$\pi\left(t_{2N-1}\right)^{*}=\pi_{N}^{*}.$$

In terms of induction method, it is easy to see that the value function always takes the form of

$$\frac{1}{1-\gamma}\left(x+y\right)^{1-\gamma}A(t),\ \ t\in\left[t_{2i},t_{2i+1}\right],$$

where $A(t)$ only depends on t. This allows us to use the Merton's strategy in the day time and to repeat the above derivation during $[t_{2i-1},\, t_{2i+1})$ for any i. The desired result then follows.

A.2 Proof of Theorem 2

Part i) can be proved using a similar argument as in Shreve and Soner (1994). To show part ii), we can follow Dai and Yi (2009) to reduce the HJB equation to a double obstacle problem in the day time (t_{2i}, t_{2i+1}). Then we can obtain $C^{2,2,1}$ smoothness of the value function for $t\in\left(t_{2i},t_{2i+1}\right)$. The smoothness of the value function in the night time is apparent. □

A.3 Proof of Proposition 1

First, let us introduce a lemma that provides connection conditions at t_{2i+1} implied by (17).

Lemma 1 *Let $z_s^*\left(t_{2i+1}\right)\in[0,\infty)$ and $z_b^*\left(t_{2i+1}\right)\in(0,\infty]$ be the sell and buy boundary at t_{2i+1} respectively. Then*

$$\begin{cases} V\left(x,y,t_{2i+1}\right)=V\left(x,y,t_{2i+1}^{+}\right), & z_s^*\left(t_{2i+1}\right)<x/y<z_b^*\left(t_{2i+1}\right) \\ -(1-\alpha)V_x\left(x,y,t_{2i+1}\right)+V_y\left(x,y,t_{2i+1}\right)=0, & x/y\le z_s^*\left(t_{2i+1}\right) \\ (1+\theta)V_x\left(x,y,t_{2i+1}\right)-V_y\left(x,y,t_{2i+1}\right)=0, & x/y\ge z_b^*\left(t_{2i+1}\right). \end{cases} \tag{A-4}$$

Proof of Lemma 1. By definition, the value function V is concave in x and y. We then deduce

$$E_b\triangleq\left\{(x,y):\,(1+\theta)V_x-V_y|_{t=t_{2i+1}^{+}}>0,\ x>0,\ y>0\right\}$$

$$E_s\triangleq\left\{(x,y):\,-(1-\alpha)V_x+V_y|_{t=t_{2i+1}^{+}}>0,\ x>0,\ y>0\right\}$$

must be connected. Here we confine ourselves to $x > 0$ and $y > 0$, in order to ensure solvency. Due to the homogeneity of the value function, we can express $z_b^*(t_{2i+1})$ and $z_s^*(t_{2i+1})$ as

$$z_b^*(t_{2i+1}) \triangleq \sup\left\{\frac{x}{y} : (x,y) \in E_b\right\},$$
$$z_s^*(t_{2i+1}) \triangleq \inf\left\{\frac{x}{y} : (x,y) \in E_s\right\}.$$

Note that for $\Delta > 0$,

$$\begin{aligned}
\frac{d}{d\Delta} V\left(x - (1+\theta)\Delta, y + \Delta, t_{2i+1}^+\right) &= -(1+\theta) V_x + V_y, \\
\frac{d}{d\Delta} V\left(x + (1-\alpha)\Delta, y - \Delta, t_{2i+1}^+\right) &= (1-\alpha) V_x - V_y.
\end{aligned}$$

Combining with (17), we get the desired result. □

By transformation (20), equations (15), (16), and (18) reduce to

$$\begin{cases}
\max\{\varphi_t + \mathcal{L}_1\varphi, (z+1-\alpha)\varphi_z - (1-\gamma)\varphi, \\
\qquad -(z+1+\theta)\varphi_z + (1-\gamma)\varphi\} = 0, \quad t \in [t_{2i}, t_{2i+1}) \\
\varphi_t + \mathcal{L}_1\varphi = 0, \qquad t \in (t_{2i-1}, t_{2i}) \\
\varphi(z,T) = \frac{1}{1-\gamma}(z+1-\alpha)^{1-\gamma},
\end{cases} \tag{A-5}$$

where

$$\mathcal{L}_1\varphi = \frac{1}{2}\sigma(t)^2 z^2 \varphi_{zz} + \beta_2(t) z \varphi_z + \beta_1(t)\varphi,$$

with $\beta_1(t) = (1-\gamma)\left(\mu(t) - \frac{1}{2}\gamma\sigma(t)^2\right)$ and $\beta_2(t) = -\left(\mu(t) - r - \gamma\sigma(t)^2\right)$. The solvency region in trading periods becomes $(-(1-\alpha), \infty) \times [0,T) \equiv \mathcal{S}_z$ in the space for the ratio z. Thanks to Lemma 1, the connection conditions at t_{2i+1} become

$$\begin{cases}
\varphi(z, t_{2i+1}) = \varphi\left(z, t_{2i+1}^+\right), & z_s^*(t_{2i+1}) < z < z_b^*(t_{2i+1}) \\
-(z+1-\alpha)\varphi_z(z, t_{2i+1}) + (1-\gamma)\varphi(z, t_{2i+1}) = 0, & z \le z_s^*(t_{2i+1}) \\
(z+1+\theta)\varphi_z(z, t_{2i+1}) - (1-\gamma)\varphi(z, t_{2i+1}) = 0, & z \ge z_b^*(t_{2i+1}).
\end{cases} \tag{A-6}$$

We further make a transformation

$$w(z,t) = \frac{1}{\gamma}\log(\gamma\varphi).$$

It follows

$$\begin{cases} \min\left\{-w_t - \mathcal{L}_2 w, \frac{1}{z+1-\alpha} - w_z, w_z - \frac{1}{z+1+\theta}\right\} = 0, & t \in [t_{2i}, t_{2i+1}) \\ -w_t - \mathcal{L}_2 w = 0, & t \in (t_{2i-1}, t_{2i}) \\ w\left(z, T\right) = \log\left(z+1-\alpha\right) \end{cases} \tag{A-7}$$

with the connection condition

$$\begin{cases} w\left(z, t_{2i+1}\right) = w\left(z, t_{2i+1}^{+}\right), & z_s^*\left(t_{2i+1}\right) < z < z_b^*\left(t_{2i+1}\right) \\ w_z\left(z, t_{2i+1}\right) = \frac{1}{z+1-\alpha}, & z \leq z_s^*\left(t_{2i+1}\right) \\ w_z\left(z, t_{2i+1}\right) = \frac{1}{z+1+\theta}, & z \geq z_b^*\left(t_{2i+1}\right). \end{cases} \tag{A-8}$$

Denote $v = w_z$. Note that

$$\begin{aligned} & \frac{\partial}{\partial z}\left(\mathcal{L}_2 w\right) \triangleq \mathcal{L} v \\ = \;& \frac{1}{2}\sigma^2(t) z^2 v_{zz} - \left(\mu(t) - r - (1+\gamma)\sigma^2(t)\right) z v_z \\ & -(\mu(t) - r - \gamma\sigma^2(t)) v + (1-\gamma)\,\sigma^2(t)\left(z^2 v v_z + z v^2\right). \end{aligned}$$

Following Dai and Yi (2009), we are able to show that v satisfies the following parabolic double obstacle problem:

$$\begin{cases} \max\left\{\min\left\{-v_t - \mathcal{L} v, v - \frac{1}{z+1+\theta}\right\}, \frac{1}{z+1-\alpha} - v\right\} = 0, & t \in [t_{2i}, t_{2i+1}) \\ -v_t - \mathcal{L} v = 0, & t \in (t_{2i-1}, t_{2i}) \\ v(z, T) = \frac{1}{z+1-\alpha} \end{cases} \tag{A-9}$$

subject to the connection condition:

$$\begin{cases} v\left(z, t_{2i+1}\right) = v\left(z, t_{2i+1}^{+}\right), & z_s^*\left(t_{2i+1}\right) < z < z_b^*\left(t_{2i+1}\right) \\ v\left(z, t_{2i+1}\right) = \frac{1}{z+1-\alpha}, & z \leq z_s^*\left(t_{2i+1}\right) \\ v\left(z, t_{2i+1}\right) = \frac{1}{z+1+\theta}, & z \geq z_b^*\left(t_{2i+1}\right). \end{cases} \tag{A-10}$$

We then infer that for any $t \in (t_{2i}, t_{2i+1})$,

$$\begin{aligned} (\mathbf{SR})_t \triangleq \left\{z: \; v(z,t) = \frac{1}{z+1-\alpha}\right\} &= \left\{z \leq z_s^*\left(t\right)\right\}, \\ (\mathbf{BR})_t \triangleq \left\{z: \; v(z,t) = \frac{1}{z+1+\theta}\right\} &= \left\{z \geq z_b^*\left(t\right)\right\}. \end{aligned}$$

Thanks to (A-9), we have

$$\left(-\frac{\partial}{\partial t} - \mathcal{L}\right)\left(\frac{1}{z+1-\alpha}\right) \leq 0 \text{ for } z \in (\mathbf{SR})_t \text{ (i.e. } z \leq z_s^*(t)), \tag{A-11}$$

$$\left(-\frac{\partial}{\partial t} - \mathcal{L}\right)\left(\frac{1}{z+1+\theta}\right) \geq 0 \text{ for } z \in (\mathbf{BR})_t \text{ (i.e. } z \geq z_b^*(t)). \tag{A-12}$$

Note that

$$\left(-\frac{\partial}{\partial t}-\mathcal{L}\right)\left(\frac{1}{z+1-\alpha}\right) = -\mathcal{L}\left(\frac{1}{z+1-\alpha}\right)$$
$$= \frac{(1-\alpha)(\mu_d-r)}{(z+1-\alpha)^3}\left[z+(1-\alpha)\frac{\mu_d-r-\gamma\sigma_d^2}{\mu_d-r}\right]$$
$$= \frac{(1-\alpha)(\mu_d-r)}{(z+1-\alpha)^3}\left[z-(1-\alpha)z_M\right] \tag{A-13}$$

and similarly

$$\left(-\frac{\partial}{\partial t}-\mathcal{L}\right)\left(\frac{1}{z+1-\mu}\right) = \frac{(1+\theta)(\mu_d-r)}{(z+1+\theta)^3}\left[z-(1+\theta)z_M\right]. \tag{A-14}$$

Combination of (A-11)-(A-14) yields the desired results.

A.4 Proof of Proposition 2

We only prove (23) as an example. First, let us show

$$z_s^*\left(t_{2i+1}^-\right) \le z_s^*(t_{2i+1}).$$

Suppose not, i.e. $z_s^*(t_{2i+1}^-) > z_s^*(t_{2i+1})$. Let $w(z,t)$ be the solution to the problem (A-7). Since $(z_s^*(t_{2i+1}^-), t_{2i+1})$ is in the no-transaction region, $w(z,t)$ is continuous at $(z_s^*(t_{2i+1}^-), t_{2i+1})$, namely, $w(z_s^*(t_{2i+1}^-), t_{2i+1}^-) = w(z_s^*(t_{2i+1}^-), t_{2i+1})$, then for $z \in (z_s^*(t_{2i+1}), z_s^*(t_{2i+1}^-))$

$$w(z,t_{2i+1}^-) = w(z_s^*(t_{2i+1}^-), t_{2i+1}^-) - \int_z^{z_s^*(t_{2i+1}^-)} \frac{1}{\xi+1-\alpha}d\xi$$
$$< w(z_s^*(t_{2i+1}^-), t_{2i+1}) - \int_z^{z_s^*(t_{2i+1}^-)} w_z(\xi, t_{2i+1})d\xi$$
$$= w(z, t_{2i+1}),$$

which contradicts the connection condition (A-8).

Clearly $z_s^*\left(t_{2i+1}^-\right) \le (1-\alpha) z_M$. So we deduce that

$$z_s^*\left(t_{2i+1}^-\right) \le \min\left\{z_s^*(t_{2i+1}), (1-\alpha) z_M\right\}.$$

If $z_s^*\left(t_{2i}^-\right) < \min\left\{z_s^*(t_{2i+1}), (1-\alpha)\, z_M\right\}$, then for

$$z \in \left(z_s^*\left(t_{2i+1}^-\right), \min\left\{z_s^*(t_{2i+1}),\ (1-\alpha)\, z_M\right\}\right),$$

we have $v\left(z, t_{2i+1}\right) = \frac{1}{z+1-\alpha}$ and

$$-v_t - \mathcal{L}v|_{(z,t_{2i+1})} = 0.$$

It follows that

$$\begin{aligned} v_t|_{(z,t_{2i+1})} &= -\mathcal{L}\left(\frac{1}{z+1-\alpha}\right) \\ &= \frac{(1-\alpha)\left(\mu_d - r\right)}{(z+1-\alpha)^3}\left[z - (1-\alpha)\, z_M\right] < 0, \end{aligned}$$

which conflicts with the fact $v_t|_{(z,t_{2i+1})} \geq 0$. The proof is complete.

A.5 Numerical procedure

The combination of (A-5) and (A-6) provides the exact model for portfolio choice with market closure. To implement the numerical procedure, we use an alternative approximation of (A-5) and (A-6), which adjusts the model during nighttime by allowing transaction but with huge transaction costs.

Thus the model for implementation of numerical procedure is

$$\begin{cases} \max\{\varphi_t + \mathcal{L}_1\varphi, (z+1-\alpha)\varphi_z - (1-\gamma)\varphi, \\ \qquad -(z+1+\theta)\varphi_z + (1-\gamma)\varphi\} = 0, \quad t \in [t_{2i}, t_{2i+1}) \\ \\ \max\left\{\varphi_t + \mathcal{L}_1\varphi, (z+1-\alpha^N)\varphi_z - (1-\gamma)\varphi, \right. \\ \qquad \left. -(z+1+\theta^N)\varphi_z + (1-\gamma)\varphi\right\} = 0, \quad t \in (t_{2i-1}, t_{2i}) \\ \\ \varphi\left(z, t_{2i+1}\right) = \varphi\left(z, t_{2i+1}^+\right), \\ \varphi(z,T) = \frac{1}{1-\gamma}(z+1-\alpha)^{1-\gamma}, \end{cases} \tag{A-15}$$

where $\alpha^N \in [0,1)$ and $\theta^N \in [0,\infty)$ are the nighttime proportional transaction costs. In the numerical procedure, we take $\alpha^N \to 1^-$ and $\theta^N \gg 1$, which makes the trading boundaries occur very close to the borders of solvency region. In another word, the sell boundary $z_s^*(t) \approx 0$ and the buy boundary $z_b^*(t) \approx \infty$ for $t \in (t_{2i-1}, t_{2i})$. In this way, trading will hardly happen during nighttime such that (A-15) is equivalent to (A-5) and (A-6) in the limit sense.

(A-15) can be numerically solved by using the penalty method with finite difference discretization developed in Dai and Zhong (2010).

References

Amihud, Y., and H. Mendelson, 1986, "Asset Pricing and the Bid-Ask Spread," *Journal of Financial Economics* 17, 223–249.

Carhart, M., 1997, "On Persistence in Mutual Fund Performance," *The Journal of Finance* 52, 57-82.

Constantinides, G. M., 1986, "Capital Market Equilibrium with Transaction Costs," *The Journal of Political Economy* 94, 842–862.

Dai, M., and F. Yi, 2009, "Finite Horizon Optimal Investment with Transaction Costs: A Parabolic Double Obstacle Problem," *Journal of Differential Equations* 246, 1445–1469.

Dai, M., and Y. Zhong, 2010, "Penalty Methods for Continuous-time Portfolio Selection with Proportional Transaction Costs," *Journal of Computational Finance* 13, 1–31.

Davis, M. H. A., and A. R. Norman, 1990, "Portfolio Selection with Transaction Costs," *Mathematics of Operations Research* 15, 676–713.

Eleswarapu, V. R., 1997, "Cost of Transacting and Expected Returns in the Nasdaq Market," *Journal of Finance* 52, 2113–2127.

Eleswarapu, V. R., and M. Reinganum, 1993, "The Seasonal Behavior of the Liquidity Premium in Asset Pricing, *Journal of Financial Economics* 34, 281–305.

Fama, E. F., and K. R. French, 1993, "Common Risk Factors in Stocks and Bonds," *Journal of Financial Economics* 33, 3–56.

Fleming, W. H., and H. M. Soner, 1993, Controlled Markov Processes and Viscosity Solutions, New York, Springer.

French, K., and R. Roll, 1986, "Stock Return Variances: The Arrival of Information and the Reaction of Traders." *Journal of Financial Economics* 17, 5–26.

Heaton, J., and D. J. Lucas, 1996, "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing," *Journal of Political Economy* 104, 443–487.

Hong, H., and J. Wang, 2000, "Trading and Returns under Periodic Market Closures," *Journal of Finance* 50, 297–354.

Jang, B. G., H. K. Koo, H. Liu, and M. Loewenstein, 2007, "Liquidity Premia and Transaction Costs," *Journal of Finance* 62, 2329–2366.

Liu, H., 2004, "Optimal Consumption and Investment with Transaction Costs and Multiple Risky Assets," *Journal of Finance* 59, 289–338.

Liu, H., and M. Loewenstein, 2002, "Optimal Portfolio Selection with Transaction Costs and Finite Horizons," *The Review of Financial Studies* 15, 805–835.

Lockwood, L. J. and S. C. Linn, 1990, "An Examination of Stock Market Return Volatility during Overnight and Intraday Periods, 1964–1989," *Journal of Finance* 45, 591–601.

Longstaff, F. A., 2009, "Portfolio Claustrophobia: Asset Pricing in Markets with Illiquid Assets," *American Economic Review* 99, 1119–1144.

Lynch, A., and S. Tan, 2011, "Explaining the Magnitude of Liquidity Premia: The Roles of Return Predictability, Wealth Shocks and State-dependent Transaction Costs," *Journal of Finance* 66, 1329–1368.

Merton, Robert C., 1971, "Optimum Consumption and Portfolio Rules in a Continuous-time Model," *Journal of Economic Theory* 3, 373–413.

Shreve, S. E., and H. M. Soner, 1994, "Optimal Investment and Consumption with Transaction Costs," *The Annals of Applied Probability* 4, 609–692.

Stoll, H. R., and R. E. Whaley, 1990, "Stock Market Structure and Volatility," *Review of Financial Studies* 3, 37–41.

Tsiakas, I., 2008, "Overnight Information and Stochastic Volatility: A study of European and US stock exchanges," *Journal of Banking & Finance* 32, 251–268.

Vayanos, D., 1998, "Transaction Costs and Asset Prices: A Dynamic Equilibrium Model," *Review of Financial Studies* 11, 1–58.